Fuzzy Behavioural Decision Model with Imperfect Information

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***Abstract*- Behavioral decision making is of growing interest of scientists and practitioners, economists and business people. A wide spectrum of successful behavioral theories is presented now, including Prospect theory (PT), multiple priors models, studies on altruism, trust and fairness. However, these theories are developed for perfect information, whereas real information on a decision maker's (DM) behavior and environment is imperfect and is described in natural language (NL). In this paper we suggest a fuzzy behavioral decision model with imperfect information in which a DM's behavior is described by a set of states with each state representing one principal behavior. In this model, states of nature and DM's states constitute a space of combined states and NL-described imperfect relevant information is formalized by the use of the fuzzy sets theory. The utility function in the suggested model is represented as a fuzzy Choquet-like integral. The suggested investigations show that Expected Utility, Choquet Expected Utility and PT are special cases of the suggested model. We apply the model to solve a benchmark and a real-life decision problem. The obtained results show validity of the suggested model.**

Keywords- Cognition and Reasoning; Decision Support Systems; Utility Assessment; Imperfect Information; Risk; Fuzzy Set

I. INTRODUCTION

Decision theory is a holy grail of numerous studies in management science, economics and other areas. It comprises a broad diversity of approaches to modeling behavior of a decision maker (DM) under various information frameworks. One of the approaches to decision making is the use of a utility function as a quantitative representation of preferences of a DM. The existing utility theories yield good results, but nowadays there is a need in generation of more realistic theories. The problem is that the existing theories are good for laboratory examples with simple, well-defined gambles. Despite that development of preference frameworks has passed a long way the modern preferences frameworks lack an important feature of human-like preferences: human-like preferences are vague [1] and are described in natural language (NL). Humans compare alternatives using linguistic techniques like "much better", "a little worse", "almost equivalent" etc [1, 2]. Such vague estimates cannot be handled by classical logic. The real decision-relevant information on states of nature, probabilities, outcomes and utilities is imperfect as perception-based and described in NL [1]. As Prof. Zadeh states, imperfect information is information which in one or more regards is imprecise, uncertain, incomplete, unreliable, vague or partially true [3].

The other important issue is that decision making is a behavioural process. During the development of decision theories scientists try to take into account features of human choices to make formal models closer to human decision activity. Risk issues were the first important behavioral issues considered in construction of decision methods. Three main categories of risk-related behaviors were introduced: risk aversion, risk seeking and risk neutrality. Gain-loss attitudes [4] and ambiguity attitudes [5] were revealed as other important behavioral features. Prospect theory [4] was the first theory incorporating both risk and gain-loss attitudes into a single utility model. Cumulative Prospect Theory (CPT) [6], as its development, is able to represent not only risk and gain-loss attitudes but also ambiguity attitudes. Choquet Expected Utility (CEU) is a well-known typical model which can be used both for ambiguity and risk situations. However, the first model developed for ambiguity aversion was Maximin Expected Utility (MMEU) [5]. Its generalization, α -MMEU, is able to represent both ambiguity aversion and ambiguity seeking [7]. Smooth ambiguity model [8] is a more advanced decision model for describing ambiguity attitudes.

A large stream of investigations compose parametric and non-parametric models based on such important psychological, moral and social aspects of decisions as reciprocity [9-11], altruism [9, 12], trust [10, 13].

A large area of research in modeling DMs (agents) in line with nature (environment) is mental-level models [14-16], idea of which was suggested in [17, 18]. In these models a DM is modeled by a set of states. Each state describes his/her possible decision-relevant condition and is referred to as "mental state", "state of mind" etc. In these models, they consider relations between mental state and state of environment [19-21]. In [14] a mental state and a state of nature compose a state of the whole system called a "global state". Within the scope of mental-level models there are two main research areas of a mental state modeling: internal modeling [14, 22] and implementation-independent modelling [14]. The first is based on modeling a mental state by a set of characteristics (variables) and the second is based on modeling a mental state on the base of beliefs, preferences and decision criterion [14].

Now we observe a significant progress in development of decision theories based on behavioral factors like risk, ambiguity, reciprocity etc. The question arises of how to adequately model joint influence of these factors on human choices. Due to highly constrained computational ability of human brain, independent influence of these factors can hardly be met. Humans conduct an intelligent, substantive comparison of alternatives in whole, i.e. as mixes of factors without pure partitioning of them. This implies interaction of factors in their influence on human choices. However, one of the disadvantages of the existing decisions theories is an absence of a due attention to interaction of factors. A vector of variables describing the factors is introduced into a decision model without fundamental consideration of how these factors really interact. Also, information on intensity of the factors and their interaction is rather uncertain and vague and can mainly be described qualitatively and not quantitatively. The mentioned issues are the main reasons of why humans are not completely rational but partially, or bounded rational and why the existing models based on pure mathematical formalism become inconsistent with human choices.

The necessity to take into account that humans are not fully rational was first conceptually addressed by Herbert Simon [23]. He proposed the concept of bounded rationality which reflects notable limitations of humans' knowledge and computational abilities. Despite their significant importance, the ideas of bounded rationality did not found its mathematical fundamentals to form a formal basis adequate to real decisions. The theory which can help to create such basis is the fuzzy set theory suggested by L.A. Zadeh [24, 25]. The reason for this is that fuzzy set theory deals with the formalization of linguistically (qualitatively) described imprecise or vague information and partial truth. Indeed, limitation of human knowledge, as one of the main aspects in bounded rationality, in real life results in the fact that humans use linguistic evaluations because the latter are tolerant for impreciseness and vagueness of real decision-relevant information. The other aspect – limitation of computational ability of humans – leads to the fact that humans think and reason in terms of propositions in NL, but not in terms of pure mathematical expressions. Such activity results in arriving at approximate solutions and satisfactory results but not at precise optimal solutions. This coincides with what is stated in bounded rationality ideas. In fuzzy logic, this is termed as approximate reasoning [26, 27]. Fuzzy sets theory was initially suggested for an analysis of humanistic systems where perceptions play a pivotal role. Perceptions are imprecise, they have fuzzy boundaries [28], and as a result, they are often described linguistically. Fuzzy sets theory and its successive technologies [29, 30] as tools for correct formal processing of perception-based information may help to arrive at perceptions-friendly and mathematically consistent decisions.

Necessity of considering decision making as a behavioural process of human reasoning with imperfect information is the main insight for development of new decision approaches. In this paper we suggest a fuzzy behavioural model for decision making with imperfect information. The suggested model can be implemented at two levels. The low-level implementation is fuzzy Choquet integral-based utility model for processing NL-described imperfect information on environment – states of nature, probabilities, outcomes and alternatives to arrive at linguistic preferences as a basis of the bounded rationality. The high-level implementation is a fuzzy decision model for processing NL-described imperfect information about both environmental and behavioural aspects. Behavioral decision analysis in our model is based on the suggested concept of a state of a DM. This concept is used to describe one principal DM's behaviour as a condition in which he/she may be when making choices. The DM's state is modelled as a subset of a space of variables describing behavioral determinants such as risk, ambiguity attitudes and others. We suggest considering the space of states of nature and space of DM's states as a single space of combined states, i.e. to consider Cartesian product of these two important spaces as basis for comparison of alternatives. Such framework allows for transparent analysis of decisions. In order to reflect the fact that both states of nature and DM's states cannot be precisely determined but are rather soft-constrained objective and subjective conditions, they are described as fuzzy sets. Likelihood of occurrence of each combined space as a pair consisting of one state of nature and one DM's state is to be described by fuzzy probability of their joint occurrence. This is motivated by uncertainty and complexity of dependence of DM's behaviour on environment.

Formalization of a DM's behavior by a set of states is an internal modeling approach. The latter, as opposed to external modeling (implementation-independent), is more adequate as based on a comprehensive and direct description of a DM and not on indirect description on the base of external elements (beliefs, outcomes, preferences). It is known that incorporating various attitudes and features of a DM into preferences sufficiently complicates the latter (for example, consider complicated preferences of CPT which cover gain-loss, risk and ambiguity attitude issues).

In our model, NL-described imperfect information on environment and DM's behaviour is formalized by using fuzzy sets theory. For a utility function in the suggested model we adopt the generalized fuzzy Choquet-like aggregation with respect to a fuzzy-valued bi-capacity. We show that expected utility (EU) [31], CEU [32] and CPT are special cases of the suggested approach.

The paper is organized as follows. In Section II we cover a prerequisite material underlying the suggested formal models. In Section III we present fuzzy Choquet integral-based utility model for decision making under NL-based imperfect relevant information. In Section IV we formulate the problem of behavioural decision making under imperfect information. Section V is devoted to fuzzy behavioural decision model with imperfect information. In the Section VI we provide a solution methodology for the problem stated in Section IV. In Section VII we apply the suggested fuzzy models to solve benchmark and real-life decision problems. Section VIII concludes.

II. PRELIMINARIES

Let E^n [33] be a space of all fuzzy subsets of R^n. These subsets satisfy the conditions of normality, convexity, and are upper semicontinuous with compact support. It is obvious that E^1 is the set of fuzzy numbers defined over R. Then let us denote by $E^1_{[0,1]}$ the corresponding space of fuzzy numbers defined over the unit interval $[0,1]$.

Let Ω be a nonempty set and $\tilde{\mathrm{F}}(\Omega) = \left\{ \tilde{V} \middle| \mu_{\tilde{V}} : \Omega \to [0,1] \right\}$ be the class of all fuzzy subsets of Ω.

Definition 1. *Hukuhara difference* [33]**.** Let $\tilde{X}, \tilde{Y} \in E^n$. If there exists $\tilde{Z} \in E^n$ so that $\tilde{X} = \tilde{Y} + \tilde{Z}$, then $\tilde{Z}$ is called a Hukuhara difference of $\tilde{X}$ and $\tilde{Y}$ and is denoted as $\tilde{X} -_h \tilde{Y}$.

Definition 2 [34]. A subclass $\tilde{\mathrm{F}}$ of $\tilde{\mathrm{F}}(\Omega)$ is called a fuzzy σ-algebra if it has the following properties:

(1) $\varnothing, \Omega \in \tilde{\mathrm{F}}$

(2) if $\tilde{V} \in \tilde{\mathrm{F}}$, then $\tilde{V}^c \in \tilde{\mathrm{F}}$

(3) if $\left\{ \tilde{V}_n \right\} \subset \tilde{\mathrm{F}}$, then $\bigcup_{n=1}^{\infty} \tilde{V}_n \in \tilde{\mathrm{F}}$

Definition 3. *Fuzzy functions.* In general, a fuzzy function is a mapping $\tilde{f} : \Omega \to E^n$, where Ω is a universe of discourse [33].

Definition 4 [34]. A fuzzy number-valued fuzzy measure (a (*z*) fuzzy measure) on $\tilde{\mathrm{F}}$ is a fuzzy number-valued fuzzy set function $\tilde{\eta} : \tilde{\mathrm{F}} \to E^1_{[0,+\infty]}$ with the properties:

(1) $\tilde{\eta}(\varnothing) = 0$;

(2) if $\tilde{V} \subset \tilde{W}$ then $\tilde{\eta}(\tilde{V}) \le \tilde{\eta}(\tilde{W})$;

(3) if $\tilde{V}_1 \subset \tilde{V}_2 \subset ..., \tilde{V}_n \subset ... \in \tilde{\mathrm{F}}$, then $\tilde{\eta}(\bigcup_{n=1}^{\infty} \tilde{V}_n) = \lim_{n\to\infty} \tilde{\eta}(\tilde{V}_n)$;

(4) if $\tilde{V}_1 \supset \tilde{V}_2 \supset ..., \tilde{V}_n \in \tilde{\mathrm{F}}$, and there exists n_0 so that $\tilde{\eta}(\tilde{V}_{n_0}) \ne \tilde{\infty}$, then $\tilde{\eta}(\bigcap_{n=1}^{\infty} \tilde{V}_n) = \lim_{n\to\infty} \tilde{\eta}(\tilde{V}_n)$.

Definition 5. *Absolute value of a fuzzy number* [35]**.** Absolute value $\left|\tilde{A}\right|$ of a fuzzy number $\tilde{A}$ is defined as

$$\mu_{|\tilde{A}|}(x) = \begin{cases} \max(\mu_{\tilde{A}}(x), \mu_{-\tilde{A}}(x)), & x \in R^+ \\ 0, & x \in R^- \end{cases}$$

Definition 6. A fuzzy number-valued bi-capacity on $\tilde{\mathrm{F}}^2 = \tilde{\mathrm{F}} \times \tilde{\mathrm{F}}$ is a fuzzy number-valued fuzzy set function $\tilde{\eta} : \tilde{\mathrm{F}}^2 \to E^1_{[-1,1]}$ with the properties:

(1) $\tilde{\eta}(\varnothing, \varnothing) = 0$;

(2) if $\tilde{V} \subset \tilde{V}'$ then $\tilde{\eta}(\tilde{V}, \tilde{W}) \le \tilde{\eta}(\tilde{V}', \tilde{W})$;

(3) if $\tilde{W} \subset \tilde{W}'$ then $\tilde{\eta}(\tilde{V}, \tilde{W}) \ge \tilde{\eta}(\tilde{V}, \tilde{W}')$;

(4) $\tilde{\eta}(\Omega, \varnothing) = 1$ and $\tilde{\eta}(\varnothing, \Omega) = -1$

In special case, values of a fuzzy-valued bi-capacity $\tilde{\eta}(\tilde{V}, \tilde{W})$ can be determined as the difference of values of two fuzzy-valued measures $\tilde{\eta}(\tilde{V}) - \tilde{\eta}'(\tilde{W})$, where “$-$” is defined on the base of Zadeh’s extension principle.

Linguistic lottery. In general, a linguistic lottery is a linguistic random variable $\tilde{S}$ with known linguistic probability distribution $\tilde{P}^l = \left\{\tilde{P}_1, ..., \tilde{P}_n\right\}$ and is represented as $\tilde{L} = \left(\tilde{P}_1, \tilde{s}_1; ...; \tilde{P}_n, \tilde{s}_n\right)$, where $\tilde{P}_i \in E^n_{[0,1]}$, $\tilde{s}_i \in E^n$.

This is the simplest definition of a linguistic lottery. In details, a linguistic lottery is formally defined in terms of consistency, completeness and redundancy of the related linguistic (fuzzy) probability distribution [36].

III. FUZZY CHOQUET INTEGRAL BASED DECISION MAKING WITH IMPERFECT INFORMATION

In our study, we generalize the axiomatization of CEU suggested by Schmeidler [32], which is constructed for perfect information framework. For real-world problems, relevant information and preferences are not exactly known and are described in NL. Based on this, our generalization involves the following: 1) Spaces of fuzzy sets [33] instead of a classical framework are used for modelling states of nature and outcomes 2) Fuzzy probabilities are used instead of numeric probabilities 3) Linguistic preference relation [36] (LPR) is used instead of binary logic-based preference relation 4) Fuzzy number-valued utility functions [1] are used instead of real-valued utility functions 5) Fuzzy number-valued fuzzy measure [34] is used instead of a real-valued nonadditive probability.

These aspects form fundamentally a new statement of the problem - the problem of decision making with imperfect information. This problem is characterized by second-order uncertainty, namely by fuzzy probabilities. Below we present formal framework of the considered problem.

Let $\mathsf{S} = \left\{\tilde{s}_1, ..., \tilde{s}_n\right\} \subset E^n$ be a set of fuzzy states of the nature, $\mathsf{X} = \left\{\tilde{X}_1, ..., \tilde{X}_n\right\} \subset E^n$ be a set of fuzzy outcomes, Y be a set of distributions of linguistic probabilities over X, i.e. Y is a set of fuzzy number-valued functions [36,82,20]: $\mathsf{Y} = \left\{\tilde{y} \middle| \tilde{y} : \tilde{\mathsf{X}} \to E^1_{[0,1]}\right\}$. For notational simplicity we identify X with the subset $\left\{\tilde{y} \in \mathsf{Y} \middle| \tilde{y}(\tilde{X}) = 1 \textit{ for some } \tilde{X} \in \mathsf{X}\right\}$ of Y. Denote by $\tilde{\mathsf{F}}_\mathsf{S}$ a σ-algebra of subsets of S. Denote by A_0 the set of all $\tilde{\mathsf{F}}_\mathsf{S}$-measurable fuzzy finite valued step functions from S to Y and denote by A_c the constant fuzzy functions in A_0. Let A be a convex subset of Y^S which includes A_c.

To model LPR, let’s introduce a linguistic variable “degree of preference” with term-set $\mathsf{T} = (T_1, ..., T_n)$. Terms can be labeled, for example, as “equivalence”, “high preference” etc, each described by a fuzzy number defined over [0, 1]. The fact that preference of $\tilde{f}$ against $\tilde{g}$ is described by $T_i \in \mathsf{T}$ is denoted $\tilde{f}\, T_i\, \tilde{g}$. We denote LPR $\cdot_l$ and sometimes, for simplicity, write $\tilde{f} \cdot^i_l \tilde{g}$ instead of $\tilde{f}\, T_i\, \tilde{g}$.

Definition 7 [1]. Two acts $\tilde{f}$ and $\tilde{g}$ in Y^S are said to be co-monotonic if there are no $\tilde{s}_i$ and $\tilde{s}_j$ in S, $\tilde{f}(\tilde{s}_i) \succ_l \tilde{f}(\tilde{s}_j)$ and $\tilde{g}(\tilde{s}_j) \succ_l \tilde{g}(\tilde{s}_i)$ hold.

In the suggested framework, elements of X are fuzzy outcomes; elements of Y are linguistic lotteries; elements of A are fuzzy acts; elements of S are fuzzy states of nature; and elements of $\tilde{\mathsf{F}}_\mathsf{S}$ are fuzzy events.

It is common knowledge that humans compare alternatives linguistically using evaluations as “much better”, “almost equivalent” etc. In contrast to the classical preference relation, LPR consistently expresses “degree of preference” allowing the analysis of preferences under uncertainty. Below we give axioms of the LPR $\cdot_l$ over A [1].

(i) ***Weak-order:*** (a) Completeness of LPR. Any two alternatives are comparable with respect to LPR: for all $\tilde{f}$ and $\tilde{g}$ in A: $\tilde{f} \cdot_l \tilde{g}$ or $\tilde{g} \cdot_l \tilde{f}$. (b) Transitivity. For all $\tilde{f}, \tilde{g}$ and $\tilde{h}$ in A: If $\tilde{f} \cdot_l \tilde{g}$ and $\tilde{g} \cdot_l \tilde{h}$ then $\tilde{f} \cdot_l \tilde{h}$.

(ii) ***Co-monotonic Independence***: For all pairwise co-monotonic acts $\tilde{f}, \tilde{g}$ and $\tilde{h}$ in A if $\tilde{f} \cdot_l \tilde{g}$, then $\alpha \tilde{f} + (1-\alpha)\tilde{h} \cdot_l \alpha \tilde{g} + (1-\alpha)\tilde{h}$ for all $\alpha \in (0,1)$.

(iii) ***Continuity***: For all $\tilde{f}, \tilde{g}$ and $\tilde{h}$ in A : if $\tilde{f} \succ_l \tilde{g}$ and $\tilde{g} \succ_l \tilde{h}$ then there are α and β in $(0,1)$ so that $\alpha \tilde{f} + (1-\alpha)\tilde{h} \succ_l \tilde{g} \succ_l \beta \tilde{f} + (1-\beta)\tilde{h}$.

(iv) ***Monotonicity***: For all $\tilde{f}$ and $\tilde{g}$ in $\mathcal{A}$: If $\tilde{f}(\tilde{s}) \succsim_l \tilde{g}(\tilde{s})$ on $\mathcal{S}$ then $\tilde{f} \succsim_l \tilde{g}$.

(v) ***Nondegeneracy***: Not for all $\tilde{f}, \tilde{g} \in \mathcal{A}$, $\tilde{f} \succsim_l \tilde{g}$.

The presented axioms are formulated to reflect human preferences under a mixture of fuzzy and probabilistic information. Such formulation requires the use of a fuzzy-valued utility function $\tilde{U}$ so that

$$\forall \tilde{f}, \tilde{g} \in \mathcal{A}, \tilde{f} \succsim_l \tilde{g} \Leftrightarrow \tilde{U}(\tilde{f}) \geq \tilde{U}(\tilde{g})$$

The problem of decision making with imperfect information consists in determination of an optimal $\tilde{f}^* \in \mathcal{A}$, that is, $\tilde{f}^* \in \mathcal{A}$ for which $\tilde{U}(\tilde{f}^*) = \max_{\tilde{f} \in A} \tilde{U}(\tilde{f})$.

Fuzzy utility function $\tilde{U}$ we adopt will be described as a fuzzy number-valued Choquet integral with respect to a fuzzy number-valued fuzzy measure. Fuzzy number-valued fuzzy measure can be obtained from NL-described knowledge about probability distribution over $\mathcal{S}$ expressed as $\tilde{P}^l = \tilde{p}_1/\tilde{s}_1 + \tilde{p}_2/\tilde{s}_2 + \tilde{p}_3/\tilde{s}_3$ = *small/small* + *high/medium* + *small/large*, with the understanding that a term such as *high/medium* means that the probability, that $\tilde{s}_2 \in \mathcal{S}$ is medium, is high. So, $\tilde{P}^l$ is a *linguistic (fuzzy) probability distribution*.

Below we present representation theorems for a fuzzy number-valued Choquet-integral-based utility function that represents LPR defined over the set $\mathcal{A}$ of alternatives under conditions of linguistic probability distribution $\tilde{P}^l$ over a set $\mathcal{S}$.

Theorem 1[1]. *Assume that LPR $\succsim_l$ on $\mathcal{A} = \mathcal{A}_0$ satisfies (i) weak order, (ii) continuity, (iii) co-monotonic independence, (iv) monotonicity, and (v) nondegeneracy. Then there exists a unique fuzzy number-valued fuzzy measure $\tilde{\eta}$ on $\tilde{\mathcal{F}}_{\mathcal{S}}$ and an affine fuzzy number-valued function $\tilde{u}$ on $\mathcal{Y}$ so that for all $\tilde{f}$ and $\tilde{g}$ in $\mathcal{A}$:*

$$\tilde{f} \succsim_l \tilde{g} \text{ iff } \int_{\mathcal{S}} \tilde{u}(\tilde{f}(\tilde{s}))d\tilde{\eta} \geq \int_{\mathcal{S}} \tilde{u}(\tilde{g}(\tilde{s}))d\tilde{\eta},$$

where $\tilde{u}$ is unique up to positive linear transformations.

Theorem 2[1]. *For a nonconstant affine fuzzy number-valued function $\tilde{u}$ on Y and a fuzzy number-valued fuzzy measure $\tilde{\eta}$ on $\tilde{\mathcal{F}}_{\mathcal{S}}$ a fuzzy number-valued Choquet integral $\tilde{U}(\tilde{f}) = \int_{\mathcal{S}} \tilde{u}(\tilde{f}(\tilde{s}))d\tilde{\eta}$ induces such LPR on $\mathcal{A}_0$ that satisfies conditions (i)-(v). Additionally, $\tilde{u}$ is unique up to positive linear transformations.*

The proofs of the theorems 1 and 2 are given [1].

In brief, a value of fuzzy utility function $\tilde{U}$ for action $\tilde{f}$ is determined as a fuzzy number-valued Choquet integral [1]:

$$\begin{aligned} \tilde{U}(\tilde{f}) &= \int_{\mathcal{S}} \tilde{u}(\tilde{f}(\tilde{s}))d\tilde{\eta}_{\tilde{P}^l} = \\ &= \sum_{i=1}^{n} \left(\tilde{u}(\tilde{f}(\tilde{s}_{(i)})) -_h \tilde{u}(\tilde{f}(\tilde{s}_{(i+1)})) \right) \cdot \tilde{\eta}_{\tilde{P}^l}(\mathrm{H}_{(i)}) \end{aligned} \tag{1}$$

Here $\tilde{\eta}_{\tilde{P}^l}()$ is a fuzzy number-valued fuzzy measure obtained from linguistic probability distribution over $\mathcal{S}$ [1] and $\tilde{u}(\tilde{f}(\tilde{s}))$ is a fuzzy number-valued utility function used to describe NL-based evaluations of utilities, (i) means that utilities are ranked so that $\tilde{u}(\tilde{f}(\tilde{s}_{(1)})) \geq \ldots \geq \tilde{u}(\tilde{f}(\tilde{s}_{(n)}))$, $\mathcal{H}_{(i)} = \left\{ \tilde{s}_{(1)}, \ldots, \tilde{s}_{(i)} \right\}$, $\tilde{u}(\tilde{f}_j(\tilde{s}_{(n+1)})) = 0$, and for each (i) there exists $\tilde{u}(\tilde{f}(\tilde{s}_{(i)})) -_h \tilde{u}(\tilde{f}(\tilde{s}_{(i+1)}))$. Mutliplication is realized in the sense of the Zadeh's extension principle.

The suggested fuzzy utility is closer to human reasoning in real choice problems in sense that it is able to process NL-based decision-relevant information and to arrive at DM's linguistic preferences as its natural consequence. However, this is not sufficient for modelling human choices influenced by behavioural determinants like risk attitudes, reciprocity etc. To take into account this issue we include space of behavioural determinants with its fuzzy granulation into principal DM's behaviours.

IV. STATEMENT OF THE PROBLEM

We suggest considering a DM's behaviour as a set of states with each state describing one principal DM's condition in which he/she may be when making choices (e.g. risk aversion). A state of a DM is modelled by a fuzzy set as a collection of qualitatively similar values of behavioural determinants. Such formalization is in the direction of internal modeling of DMs within the scope of mental-level models. However, in contrast to the existing works, our DM's states are not exclusive, but overlapping sets to reflect the fact that the state of a DM is uncertain itself and cannot be sharply bounded. Uncertainty related to occurence of a state of a DM is described by a linguistic (fuzzy) probability. Fuzzy probability describes impreciseness of beliefs coming from uncertainty and complexity of interaction of the factors, from absence of ideal information.

The space of states of nature and space of DM's states constitute a Cartesian product as a single space of combined states [37, 38]. Likelihood of occurrence of each combined state as a pair of one state of nature and one DM's state is described by fuzzy probability of their joint occurrence. This fuzzy joint probability (FJP) is found on the base of fuzzy marginal probabilities of state of nature and state of a DM and, if possible, on the base of some information about dependence of these states. Utilities of outcomes are also to be distributed over the combined states reflecting naturally various evaluation of the outcomes by a DM in his/her various states.

Let $\mathrm{S}=\{\tilde{s}_1,\tilde{s}_2,...,\tilde{s}_M\}\subset E^n$ be a space of fuzzy states of nature and X be a space of fuzzy outcomes as a bounded subset of E^n. Denote by $\mathrm{H}=\left\{\tilde{h}_1,\tilde{h}_2,...,\tilde{h}_N\right\}\subset E^n$ a set of fuzzy states of a DM [38]. Then we call $\Omega=\mathrm{S}\times\mathrm{H}$ a space "nature-DM", elements of which are combined states $\tilde{w}=(\tilde{s},\tilde{h})$ where $\tilde{s}\in\mathrm{S},\tilde{h}\in\mathrm{H}$. Denote $\tilde{\mathrm{F}}_{\Omega}$ a σ-algebra of subsets of Ω. Then consider $\mathrm{A}=\left\{\tilde{f}\in\mathrm{A}\,\middle|\,\tilde{f}:\Omega\to\mathrm{X}\right\}$ the set of fuzzy actions as the set of all $\tilde{\mathrm{F}}_{\Omega}$-measurable fuzzy functions from Ω to X [38].

A problem of behavioral decision making with combined states under imperfect information can be denoted as $D=(\Omega,\mathrm{X},\mathrm{A},\cdot_{l})$ where $\cdot_{l}$ are linguistic preferences of a DM.

In general, it is not known which state of nature will take place and what state of a DM will present at the moment of decision making. Only some partial knowledge on probability distributions on S and H is available. An information relevant to a DM can be formalized as a linguistic probability distribution over his/her states: $\tilde{p}_1/\tilde{h}_1+\tilde{p}_2/\tilde{h}_2+...+\tilde{p}_N/\tilde{h}_N$, where $\tilde{p}_i$ is a linguistic belief degree or a linguistic probability. So, $\tilde{p}_i/\tilde{h}_i$ can be formulated as "a probability that a DM's state is $\tilde{h}_i$ is $\tilde{p}_i$".

For closer description of human behavior and imperfect information on Ω we use a fuzzy number-valued bi-capacity $\tilde{\eta}=\tilde{\eta}(\tilde{V},\tilde{W})$, $\tilde{V},\tilde{W}\subset\Omega$. Value or utility of an outcome $\tilde{y}=\tilde{f}(\tilde{s},\tilde{h})$ in various DM's states will also be various, and then can be formalized as a function $\tilde{u}(\tilde{y})=\tilde{u}(\tilde{f}(\tilde{s},\tilde{h}))$. We can claim that the value function of Kahneman and Tversky $v=v(f(s))$ [4] appears then as a special case. An overall utility $\tilde{U}(\tilde{f})$ of an action $\tilde{f}$ is determined as a fuzzy number-valued bi-capacity-based aggregation of $\tilde{u}(\tilde{f}(\tilde{s},\tilde{h}))$ over space Ω. Then the problem consists in determination of an optimal action $\tilde{f}^*\in\mathrm{A}$ with $\tilde{U}(\tilde{f}^*)=\max\limits_{\tilde{f}\in\mathrm{A}}\int_{\Omega}\tilde{U}(\tilde{f}(\tilde{w}))d\tilde{\eta}$.

V. FUZZY BEHAVIORAL DECISION MODEL WITH IMPERFECT INFORMATION

As the basis for our model we use the framework formulated [39]. The linguistic preference means that the preference among actions $\tilde{f}$ and $\tilde{g}$ is modeled by a degree $Deg(\tilde{f}\cdot_{l}\tilde{g})\in[0,1]$ to which $\tilde{f}$ is at least as good as $\tilde{g}$ and a degree $Deg(\tilde{g}\cdot_{l}\tilde{f})$ to which $\tilde{g}$ is at least as good as $\tilde{f}$. For special case, when $Deg(\tilde{g}\cdot_{l}\tilde{f})=0$ and $Deg(\tilde{f}\cdot_{l}\tilde{g})\neq 0$ we have the classical preference, i.e. we say that $\tilde{f}$ is preferred to $\tilde{g}$.

The base for our model is composed by intra-combined state information and inter-combined states information. Intra-combined state information is used to form utilities representing preference over outcomes $\tilde{f}(\tilde{w}_i)=\tilde{x}_i$ of an act $\tilde{f}\in\mathrm{A}$ where $\tilde{w}_i=(\tilde{s}_{i_1},\tilde{h}_{i_2})$, with understanding that these are preferences at state of nature $\tilde{s}_{i_1}$ conditioned by a state $\tilde{h}_{i_2}$ of a DM. Inter-combined states information is used to form fuzzy-valued bi-capacity representing dependence between combined states as human behaviors under imperfect information.

We use bi-capacity-adopted integration [39] at the space "nature-DM" for determination of an overall utility of an alternative. The bi-capacity is a natural generalization of capacities and is able to describe interaction between attractive and repulsive values (outcomes, criteria values), particularly, gains and losses. For an overall utility $\tilde{U}$ of action $\tilde{f}$ we use a fuzzy-valued generalized Choquet-like aggregation over Ω with respect to fuzzy-valued bi-capacity $\tilde{\eta}(\cdot,\cdot)$:

$$\tilde{U}(\tilde{f})= \\ =\sum_{l=1}^{n}(\tilde{u}(\tilde{f}(\tilde{w}_{(l)}))-_h\tilde{u}(\tilde{f}(\tilde{w}_{(l+1)})))\tilde{\eta}(W_{(l)}\cap N^{+},W_{(l)}\cap N^{-}) \tag{2}$$

provided $\tilde{u}(\tilde{f}(\tilde{w}_{(l)}))\geq\tilde{u}(\tilde{f}(\tilde{w}_{(l+1)}))$; $W_{(l)}=\{\tilde{w}_{(1)},...,\tilde{w}_{(l)}\}$, $N^{+}=\{\tilde{w}\in\Omega:\tilde{u}(\tilde{f}(\tilde{w}))\geq 0\}$, $N^{-}=\Omega\setminus N^{+}$

An optimal action $\tilde{f}^{*}\in\mathrm{A}$, that is $\tilde{f}^{*}\in\mathrm{A}$ for which $\tilde{U}(\tilde{f}^{*})=\max\limits_{\tilde{f}\in\mathrm{A}}\left\{\int_{\Omega}\tilde{u}(\tilde{f}(\tilde{s},\tilde{h})d\tilde{\eta}_{\tilde{P}^l}\right\}$ is found by a determination of $Deg(\tilde{f}\cdot_l\tilde{g})$, $\tilde{f},\tilde{g}\in\mathrm{A}$: optimal action $\tilde{f}^{*}\in A$ is such an action that $Deg(\tilde{f}^{*}\cdot_l\tilde{f})\geq Deg(\tilde{f}\cdot_l\tilde{f}^{*})$ for all $\tilde{f}\in\mathrm{A},\tilde{f}\neq\tilde{f}^{*}$. The determination of $Deg(\tilde{f}\cdot_l\tilde{g})$ is based on comparison of $\tilde{U}(\tilde{f})$ and $\tilde{U}(\tilde{g})$ respectively as follows. Membership functions of $\tilde{U}(\tilde{f})$ and $\tilde{U}(\tilde{g})$ describe possibilities of various precise values $U(\tilde{f})$ and $U(\tilde{g})$ of overall utilities of $\tilde{f}$ and $\tilde{g}$. That is, there is possibility $r\in(0,1]$ that precise overall utilities of $\tilde{f}$ and $\tilde{g}$ are equal to $U_1^r(\tilde{f}),U_2^r(\tilde{f})$ and $U_1^r(\tilde{g}),U_2^r(\tilde{g})$ respectively. Therefore, there is possibility $r\in(0,1]$ that the difference between precise overall utilities of $\tilde{f}$ and $\tilde{g}$ is $U_i^r(\tilde{f})-U_j^r(\tilde{g}),i,j=1,2$. As $\tilde{f}$ is preferred to $\tilde{g}$ when overall utility of $\tilde{f}$ is larger than that of $\tilde{g}$, we consider only $U_i^r(\tilde{f})-U_i^r(\tilde{g})>0$. Consider the following functions:

$$\sigma(r)=\sum_{i=1}^{2}\sum_{j=1}^{2}\max(U_i^r(\tilde{f})-U_j^r(\tilde{g}),0);$$

$$\delta_{ij}(r)= \\ =\begin{cases}\dfrac{\max(U_i^r(\tilde{f})-U_j^r(\tilde{g}),0)}{\left|U_i^r(\tilde{f})-U_j^r(\tilde{g})\right|}, \text{ if } U_i^r(\tilde{f})-U_j^r(\tilde{g})\neq 0 \\ 0, \text{ else}\end{cases} \quad i,j=1,2$$

$$\delta(r)=\sum_{i}^{2}\sum_{j}^{2}\delta_{ij}(r).$$

$\sigma(r)$ shows the sum of all positive differences between $U_1^r(\tilde{f}),U_2^r(\tilde{f})$ and $U_1^r(\tilde{g}),U_2^r(\tilde{g})$ and $\delta(r)$ shows the number of these differences. Consider now the quantity $\dfrac{\int_0^1 r\sigma(r)dr}{\int_0^1 r\delta(r)dr}$ as a weighted average of differences $U_i^r(\tilde{f})-U_j^r(\tilde{g}),\ i,j=1,2$ where weights are their possibilities $r\in(0,1]$. The degree $Deg(\tilde{f}\succ_l\tilde{g})$ is determined then as follows:

$$Deg(\tilde{f} \cdot_l \tilde{g}) = \frac{\int_0^1 r\sigma(r)dr}{(u_{\max} - u_{\min})\int_0^1 r\delta(r)dr} \tag{3}$$

In other words, $Deg(\tilde{f} \cdot_l \tilde{g})$ is a percentage of a weighted average of $U_i^r(\tilde{f}) - U_j^r(\tilde{g}),\ i, j = 1, 2$ with respect to maximally possible difference $u_{\max} - u_{\min}$ (as for all $\tilde{f}, \tilde{g} \in A,\ U_i^r(\tilde{f}) - U_j^r(\tilde{g}) \le u_{\max} - u_{\min};\ i, j = 1, 2$ is satisfied). The closer the $U_i^r(\tilde{f}) - U_j^r(\tilde{g})$ to $u_{\max} - u_{\min}$ is, the higher the extent to which $\tilde{f}$ is better than $\tilde{g}$ is.

Let us show that the famous utility models are special cases of the proposed combined states-based fuzzy utility model. To do this, we simplify our model to its non-fuzzy variant. Bi-capacity-based aggregation of $u(f(s,h))$ on a space Ω would be a natural generalization of an aggregation of $u(f(s))$ on a space S. We will show this by comparing EU and CEU applied on space S with the same models applied on a combined states space Ω. For obvious illustration let us look at a general graphic representation of combined states space Ω given in Table 1:

TABLE 1 COMBINED STATES SPACE

	s_1	$\cdots$	s_i	$\cdots$	s_n
h_1	(s_1, h_1)	$\cdots$	(s_i, h_1)	$\cdots$	(s_n, h_1)
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
h_j	(s_1, h_j)	$\cdots$	(s_i, h_j)	$\cdots$	(s_n, h_j)
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
h_m	(s_1, h_1)	$\cdots$	(s_i, h_m)	$\cdots$	(s_n, h_m)

EU used for combined states space (Table 1) will have the following form:

$$U(f) = \sum_{k=1}^{mn} u(f(w_k))p(w_k) = \sum_{j=1}^{m}\sum_{i=1}^{n} u(f(s_i, h_j))p(s_i, h_j) \tag{4}$$

In traditional EU (EU applied to a space S only) they consider that a DM exhibits the same behavior in any state of nature. In our approach this means that only one state of a DM exists. Then, to model a classical EU within (4) we should exclude all h_j except one, say, h_k. This means that $p(s_i, h_j) = 0, \forall j \ne k$ (as we consider that all $h_j,\ j \ne k$ don't exist) and we have

$$U(f) = \sum_{i=1}^{n} u(f(s_i, h_k))p(s_i, h_k)$$

Now, as a DM is always at a state h_k whatever state s_i takes place, we have $p(s_i, h_k) = p(s_i)$. Furthermore, in traditional EU only risk attitudes as behavioral aspects are taken into account. A DM is considered as either risk averse or risk seeking or risk neutral. So, h_k can represent one of these behaviors. For example, if h_k represents risk aversion, then $u()$ will be concave, if h_k represents risk seeking then $u()$ will be convex etc. If we use notation $u^*(f(\cdot))$ for $u(f(\cdot, h_k))$ when h_k represents, for example, risk aversion, we have (4) as $U(f) = \sum_{i=1}^{n} u^*(f(s_i))p(s_i)$ which is nothing but a traditional EU. So, traditional EU is a special case of the EU used for Ω. Combined-states based approach as opposed to traditional EU allows to take into account that a DM can exhibit various risk attitudes at various states of nature. This usually takes place in real life and is taken into account in PT and CPT (these models are based on experimental observations demonstrating that people exhibit risk aversion for gains and risk seeking for losses).

Let us now show that CEU used for space S is a special case of CEU used for Ω. CEU used for Ω has the following form:

$$U(f)=\sum_{l=1}^{n}(u(f(w_{(l)}))-u(f(w_{(l+1)})))\eta(\{w_{(1)},...,w_{(l)}\}) \tag{4}$$

$w_{(l)}=(s_j,h_k)$, $u(f(w_{(l)}))\geq u(f(w_{(l+1)}))$. Assuming now that only some h_k exists, we have that $\forall w\in\Omega, w=(s_i,h_k)$, that is $\Omega=\mathrm{S}\times\{\tilde{h}_k\}$. Then we will have $u(f(w_{(l)}))-u(f(w_{(l+1)}))=0$ whenever $w_{(l)}=(s_i,h_k), w_{(l+1)}=(s_i,h_k)$. Only differences $u(f(w_{(l)}))-u(f(w_{(l+1)}))$ for which $w_{(l)}=(s_i,h_k), w_{(l+1)}=(s_j,h_k), i\neq j$ may not be equal to zero. As a result, making simple transformations, we will have:

$U(f)=\sum_{l=1}^{n}(u(f(w_{(l)}))-u(f(w_{(l+1)})))\eta(\{w_{(1)},...,w_{(l)}\})$ $w_{(l)}=(s_j,h_k)$, $u(f(w_{(l)}))\geq u(f(w_{(l+1)}))$. Now, using notations $u^*(f(\cdot))$ for $u(f(\cdot,h_k))$ and $\eta^*(\{s_{(1)},...,s_{(j)}\})=\eta(\{(s_{(1)},h_k),...,(s_{(j)},h_k)\})$ we can write

$$U(f)=\sum_{i=1}^{n}(u^*(f(s_{(i)}))-u^*(f(s_{(i+1)})))\eta^*(\{s_{(1)},...,s_{(i)}\})$$

This is nothing but a traditional CEU. Traditional CEU is often used to represent uncertainty attitude as an important behavioral aspect. So, if h_k represents uncertainty aversion (uncertainty seeking) then $\eta(\{(s_{(1)},h_k),...,(s_{(j)},h_k)\})$ can be chosen as lower prevision (upper prevision).

It can also be shown that the model used in the CPT is also a special case of the combined states approach. This follows from the fact that representation used in CPT is a sum of two Choquet integrals.

VI. SOLUTION OF THE PROBLEM

The solution of the problem consists in determination of an optimal action $\tilde{f}^*\in\mathrm{A}$ with $\tilde{U}(\tilde{f}^*)=\max_{\tilde{f}\in\mathrm{A}}\left\{\int_{\Omega}\tilde{u}(\tilde{f}(\tilde{s},\tilde{h}))d\tilde{\eta}\right\}$. The problem is solved as follows. At the first stage linguistic utility values $\tilde{u}(\tilde{f}(\tilde{s}_i,\tilde{h}_j))$ are assigned to every action $\tilde{f}\in\mathrm{A}$ taken at a state of nature $\tilde{s}_i\in\mathrm{S}$ when a DM's state is $\tilde{h}_j$. The second stage consists in construction of a FJP distribution $\tilde{P}^l$ on Ω proceeding from partial information on marginal distributions over S and H which is represented by given fuzzy probabilities for all states except one. This requires constructing unknown fuzzy probability for each space [1]. Given marginal distribution of fuzzy probabilities for all the states, it is needed to verify consistency, completeness and redundancy of this distribution [37,38]. Finally, on the base of fuzzy marginal distributions (for S and H) and information on dependence between states of nature $s\in\mathrm{S}$ and a DM's states $h\in\mathrm{H}$ it is needed to construct FJP distribution $\tilde{P}^l$ on Ω.

At the third stage it is necessary to construct a fuzzy-valued bi-capacity $\tilde{\eta}(\cdot,\cdot)$ based on FJP $\tilde{P}^l$ on Ω. Next the problem of calculation of an overall utility $\tilde{U}(\tilde{f})$ for every action $\tilde{f}\in A$ is solved by using Formula (1). In (1) differences between fuzzy utilities $\tilde{u}(\tilde{f}(\tilde{s},\tilde{h}))$ assigned at the first stage are multiplied on the base of the Zadeh's extension principle by the values of the fuzzy valued bi-capacity $\tilde{\eta}(\cdot,\cdot)$ constructed at the third stage.

Finally, an optimal action $\tilde{f}^*\in\mathrm{A}$ as the action with the maximal fuzzy valued utility $\tilde{U}(\tilde{f}^*)=\max_{\tilde{f}\in\mathrm{A}}\left\{\int_{\Omega}\tilde{u}(\tilde{f}(\tilde{s},\tilde{h}))d\tilde{\eta}\right\}$ is determined by comparing fuzzy overall utilities $\tilde{U}(\tilde{f})$ for all $\tilde{f}\in\mathrm{A}$ (see Formula (2)).

VII. EXAMPLES

A. Example 1. Business Development for a Computer Firm

An owner of a computer firm needs to evaluate results of the possible actions on his business development over the next five years. There has been good sales growth over the past couple of years. The owner sees three options. The first is to enlarge the current store, the second is to move to a new site, and the third is to simply wait and do nothing. The action on expand or on move would take little time, and, so, the store would not lose revenue. Waiting longer than one year would allow competition to move in and would make expansion no longer feasible.

The NL-based description of assumptions and conditions with linguistically described values of monetary outcomes and probabilities for the problem are as follows:

"*Strong growth* as a result of the increased population of computer buyers from the electronics new firm has "*a little higher than medium*" probability. Moving to a new location under *strong growth* will result in a "*significant*" outcome. Moving to a new location under *weak growth* will result in an "*about medium*" outcome. Expanding store under *strong growth* will result in a "*large*" outcome, whereas expanding store under *weak growth* will result in a "*medium*" outcome. Doing nothing under *strong growth* will result in an "*about large*" outcome and doing noting under *strong growth* will result in a "*higher than medium*" outcome."

Formal description of the problem. The set of the fuzzy states of the nature is $S = \{\tilde{s}_1, \tilde{s}_2\}$, where $\tilde{s}_1$ - "strong growth", $\tilde{s}_2$ - "weak growth". The membership functions of $\tilde{s}_1$ and $\tilde{s}_2$ defined over the scaled domain are shown in the Fig. 1:

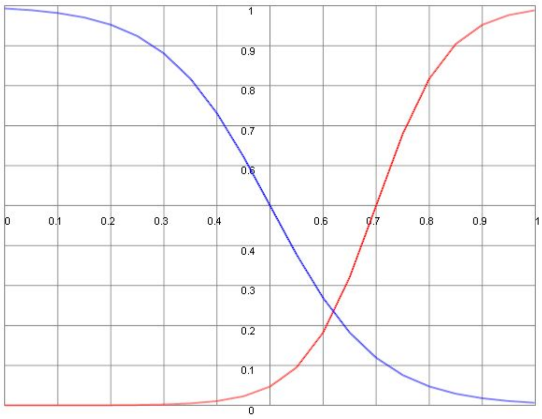

Fig. 1 Fuzzy states of nature

The set of the manager's possible actions is $A = \{\tilde{f}_1, \tilde{f}_2, \tilde{f}_3\}$, where $\tilde{f}_1$ is "move" decision, $\tilde{f}_2$ is "expand" decision, and $\tilde{f}_3$ is "do nothing" decision. The set of outcomes is $Y = \{\tilde{f}_1(\tilde{s}_1), \tilde{f}_1(\tilde{s}_2), \tilde{f}_2(\tilde{s}_1), \tilde{f}_2(\tilde{s}_2), \tilde{f}_3(\tilde{s}_1), \tilde{f}_3(\tilde{s}_2)\}$. Let fuzzy values of monetary outcomes be described as triangular fuzzy numbers as follows: "about medium" = (286.5; 365; 401.5); "medium" = (354.3; 413; 454.3); "higher than medium" = (472.5; 525; 577.5); "significant" = (646.5; 765; 841.5); "about large"=(760; 850; 930); "large"= (759.3; 863; 949.3). The linguistic probability distribution $\tilde{P}^l$ over the states of nature that corresponds the knowledge the manager has is:

$$\tilde{P}^l = \tilde{P}_1 / \tilde{s}_1 + \tilde{P}_2 / \tilde{s}_2$$

$\tilde{P}_1 =$ "a little higher than medium", described by triangular fuzzy number $(0.45; 0.55; 0.65)$ and $\tilde{P}_2$ is unknown. $\tilde{P}_1$ (given) and $\tilde{P}_2$ (obtained) probability values are given in the Fig. 2:

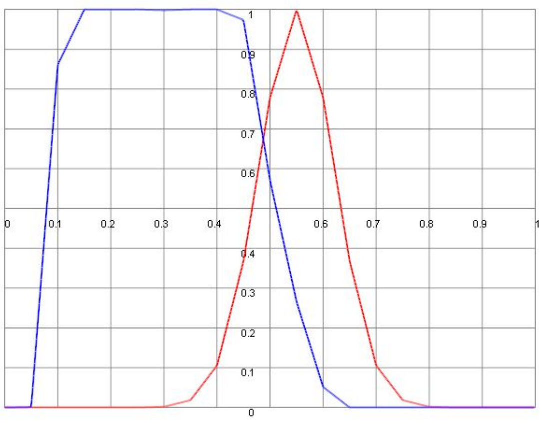

Fig. 2 The given (blue curve) and the obtained (red curve) fuzzy probabilities

The fuzzy number-valued fuzzy measure $\tilde{\eta}_{\tilde{P}^l}$ obtained from $\tilde{P}^l$ and approximated by triangular fuzzy numbers is given in Table 2:

TABLE 2 FUZZY NUMBER-VALUED FUZZY MEASURE $\tilde{\eta}_{\tilde{P}^l}$

H	$\{\tilde{s}_1\}$	$\{\tilde{s}_2\}$	$\{\tilde{s}_1,\tilde{s}_2\}$
$\tilde{\eta}_{\tilde{P}^l}$ (H)	(0.4, 0.57, 0.57)	(0.27, 0.45,0.45)	1

Fuzzy utility values for $\tilde{f}_j, j=\overline{1,3}$ are calculated as follows:

$$\tilde{U}(\tilde{f}_j)=(\tilde{u}(\tilde{f}_j(\tilde{s}_{(1)}))-_h \tilde{u}(\tilde{f}_j(\tilde{s}_{(2)})))\times\tilde{\eta}_{\tilde{P}^l}(\{\tilde{s}_{(1)}\})+\tilde{u}(\tilde{f}_j(\tilde{s}_{(2)}))\tilde{\eta}_{\tilde{P}^l}(\{\tilde{s}_1,\tilde{s}_2\})$$

For simplicity we define fuzzy utilities $\tilde{u}(\tilde{f}_j(\tilde{s}_{(i)}))$ to be numerically equal to fuzzy values of outcomes $\tilde{f}_j(\tilde{s}_{(i)})$.

The calculated fuzzy overall utilities $\tilde{U}(\tilde{f}_j), j=\overline{1,3}$ for $\tilde{f}_j, j=\overline{1,3}$ approximated as TFNs are the following:

$$\tilde{U}(\tilde{f}_1)=(383.7;\ 545;\ 599.5);$$

$$\tilde{U}(\tilde{f}_2)=(463.65;\ 615.5;\ 677.05);$$

$$\tilde{U}(\tilde{f}_3)=(550.125;\ 671.25;\ 736.125).$$

These fuzzy utility values are shown in Fig. 3:

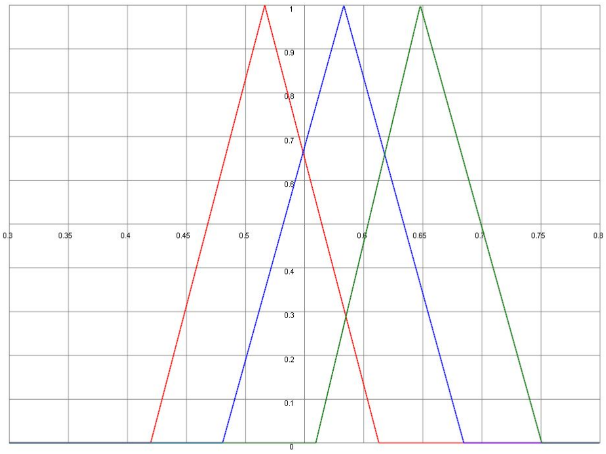

Fig. 3 Fuzzy utility values of the manager's possible actions (for $\tilde{f}_1$ -doted line, for $\tilde{f}_2$ - dashed line, for $\tilde{f}_3$ -solid line)

The comparison of this fuzzy numbers allows to determine the best alternative as an alternative with the highest fuzzy utility.

B. Bingo Game

Let us conduct comparison of the combined states-based approach and CPT for solving a benchmark decision problem – the famous Bingo game [40]. The considered behavioral aspects are risky issues.

Let a transparent Bingo Blower box contain red, blue and green balls numbers of which cannot be visually counted but can only be defined by means of imprecise linguistic evaluations as follows: "the number of blue balls is medium, the number of red balls is about thirty percents and the number of green balls is the least". Assume two alternative games:

$100 is given if a green ball is extracted, $10 is given if a blue ball is extracted and $10 is taken if a red ball is extracted;

$10 is given if a green ball is extracted, $10 is taken if a blue ball is extracted and $100 is given if a red ball is extracted.

The set of states of nature is $S=\{s_1,s_2,s_3\}$, where s_1, s_2, s_3 – extractions of a blue, a red, and a green ball respectively. To precisiate linguistic information on probabilities of the states of nature we use triangular fuzzy numbers (TFNs):

$$\tilde{p}(s_1)=(0.45,0.5,0.55),\quad \tilde{p}(s_2)=(0.25,0.3,0.35),\quad \tilde{p}(s_3)=(0.15,0.2,0.25)$$

So, we have the following alternatives:

$$f_1=(100;10;-10),\ f_2=(10;-10;100).$$

Let us reveal preferences over these alternatives by applying combined states-based approach as a generalization of CPT. According to CPT, a DM is risk averse for gains and risk seeking for losses. Thus, we model a DM behaviour by a set of states $H=\{h_1,h_2\}$, where h_1 and h_2 (for simplicity, considered binary – non-fuzzy) are the DM's states representing risk aversion and risk seeking respectively. Then the ideas underlying CPT can be realized as follows. For f_1 there may be only the following combined states: $(s_1,h_1),(s_2,h_1),(s_3,h_2)$. Indeed, in s_1 and s_2 a gain takes place, and therefore, a DM is risk averse, that is a DM's state is h_1; in s_3 a loss takes place and then a DM is a risk seeking, that is a DM's state is h_2. Any other combined state, for example (s_1,h_2) is impossible – in CPT they do not consider a DM risk seeking in domain of gains. However, these ideas should not be considered as dogma. A DM, who is risky in his character, would likely exhibit risk seeking also in domain of gains. There may also be other evidence. Combined states space Ω as all combinations of states of nature and DM's states are shown in Table 3:

TABLE 3 COMBINED STATES FOR THE RISK ATTITUDES

	s_1	s_2	s_3
h_1	(s_1,h_1)	(s_2,h_1)	(s_3,h_1)
h_2	(s_1,h_2)	(s_2,h_2)	(s_3,h_2)

We determine an overall utility of an alternative as Choquet-like bi-capacity-based aggregation of its utilities $u(f(s_i,h_j))$ $(f\in\{f_1,f_2\};i=1,2,3;\ j=1,2)$ over Ω with respect to a fuzzy-valued bi-capacity $\tilde{\eta}$. $u(f(\cdot,h_j))$ will be chosen concave for h_1 and convex for h_2 by applying, for simplicity, the technique adopted to construct a value function in CPT:

$$u(f(s_i,h_1))=\begin{cases}(f(s_i))^{\alpha},\ f(s_i)\ge 0\\ -\lambda(-f(s_i))^{\beta},\ f(s_i)<0\end{cases}$$
$$u(f(s_i,h_2))=\begin{cases}(f(s_i))^{\beta},\ f(s_i)\ge 0\\ -\lambda(-f(s_i))^{\alpha},\ f(s_i)<0\end{cases} \tag{5}$$

$\alpha=0.88,\ \beta=1.25$. $\tilde{\eta}$ will be obtained from FJP $\tilde{p}(s,h)$ of combined states $(s,h),\ s\in S,h\in H$. Next, according to Section 3 we define FJP $\tilde{p}(s,h)$ given marginal probabilities $\tilde{p}(s_i)$ and $\tilde{p}(h_j)$. We have found that under fuzzy marginal probability distribution $\tilde{p}(\tilde{h}_1)=(0.55,0.6,0.65)$, $\tilde{p}(\tilde{h}_2)=(0.35,0.4,0.45)$ over H (a DM is more likely to be risk averse) we obtain the same preferences over our alternatives as those obtained by CPT. Below we describe the way we obtained the preferences. Given $\tilde{p}(\tilde{s})$, $\tilde{p}(\tilde{h})$ we calculate FJP $\tilde{p}(s,h)$ on the base of concepts of positive dependence and negative dependence [41, 42] or f_1 we suppose positive dependence between states in the pairs $(s_1,h_1),(s_2,h_1),(s_3,h_2)$ (as these are combinations of states nature and DM's behaviors that take place according to CPT) and negative dependence between states $(s_1,h_2),(s_2,h_2),(s_3,h_1)$ (as these are combinations of states nature of nature and a DM's behavior that do not take place according to CPT). Then the FJPs are obtained as the following trapezoidal fuzzy numbers: $\tilde{p}(s_1,h_1)=(0.083,0.12,0.2,0.25)$, $\tilde{p}(s_2,h_1)=(0.2475,0.3,0.5,0.55)$, $\tilde{p}(s_3,h_1)=(0,0,0.18,0.26)$, $\tilde{p}(s_1,h_2)=(0,0,0.08,0.1125)$, $\tilde{p}(s_2,h_2)=(0,0,0.2,0.2475)$, $\tilde{p}(s_3,h_2)=(0.07,0.12,0.3,0.4)$. For f_2, supposing positive dependence between states in $(s_1,h_1),(s_2,h_2),(s_3,h_1)$ and negative dependence between states in $(s_1,h_2),(s_2,h_1),(s_3,h_2)$, we have: $\tilde{p}(s_1,h_1)=(0.0825,0.12,0.2,0.25)$, $\tilde{p}(s_2,h_1)=(0.1,0.2,0.3,0.3575)$, $\tilde{p}(s_3,h_1)=(0.11,0.18,0.3,0.4)$, $\tilde{p}(s_1,h_2)=(0,0,0.08,0.1125)$, $\tilde{p}(s_2,h_2)=(0.1575,0.2,0.4,0.45)$, $\tilde{p}(s_3,h_2)=(0,0,0.12,0.18)$.

Next we obtain fuzzy overall utilities $\tilde{U}(f_i), i=1,2$. For example, $\tilde{U}(f_1)$ is expressed as follows:

$$\begin{aligned}
&\tilde{U}(f_1) = (|u(f_1(s_1,h_2))| - |u(f_1(s_1,h_1))|)\times \\
&\times(\tilde{\eta}(\{(s_1,h_2)\},\varnothing)) + \\
&(|u(f(s_1,h_1))| - |u(f(s_3,h_1))|)\times \\
&\times(\tilde{\eta}(\{(s_1,h_2),(s_1,h_1)\},\varnothing)) + \\
&(|u(f(s_3,h_1))| - |u(f(s_2,h_2))|)\times \\
&\times\tilde{\eta}(\{(s_1,h_2),(s_1,h_1)\},\{(s_3,h_1)\}) + \\
&(|u(f(s_2,h_2))| - |u(f(s_3,h_2))|)\times \\
&\times\tilde{\eta}(\{(s_1,h_2),(s_1,h_1),(s_2,h_2)\},\{(s_3,h_1)\}) + \\
&(|u(f(s_3,h_2))| - |u(f(s_2,h_1))|)\times \\
&\times\tilde{\eta}(\{(s_1,h_2),(s_1,h_1),(s_2,h_2)\},\{(s_3,h_1),(s_3,h_2)\}) + \\
&|u(f(s_2,h_1))|\times \\
&\times\tilde{\eta}(\{(s_1,h_2),(s_1,h_1),(s_2,h_2),(s_2,h_1)\},\{(s_3,h_1),(s_3,h_2)\})
\end{aligned} \tag{6}$$

where $u(f(\cdot,\cdot))$ utilities are determined by (5). We define fuzzy-valued bi-capacity $\tilde{\eta}_{\tilde{P}^l}(\cdot,\cdot)$ as a difference $\tilde{\eta}_{\tilde{P}^l}(V,W) = \tilde{\eta}_{\tilde{P}^l}(V) - \tilde{\eta}_{\tilde{P}^l}(W)$ of values of a fuzzy-valued lower probability $\tilde{\eta}_{\tilde{P}^l}$. The values of $\tilde{\eta}_{\tilde{P}^l}(V,W)$ for f_1 are given in Table 4:

TABLE 4 FUZZY-VALUED BI-CAPACITY OBTAINED FOR f_1

$V,W \subset \Omega$	$\tilde{\eta}_{\tilde{P}^l}(V,W)$
$\{(s_1,h_2)\},\varnothing$	(0,0,0)
$\{(s_1,h_2),(s_1,h_1)\},\varnothing$	(0,0.0825,0.12)
$\{(s_1,h_2),(s_1,h_1)\},\{(s_3,h_1)\}$	(0,0.0825,0.12)
$\{(s_1,h_2),(s_1,h_1),(s_2,h_2)\},\{(s_3,h_1)\}$	(0,0.0825,0.12)
$\{(s_1,h_2),(s_1,h_1),(s_2,h_2)\},\{(s_3,h_1),(s_3,h_2)\}$	(0,0.0825,0.12)
$\{(s_1,h_2),(s_1,h_1),(s_2,h_2),(s_2,h_1)\},\{(s_3,h_1),(s_3,h_2)\}$	(0.22,0.3,0.35)

The overall fuzzy utilities obtained as TFNs are $\tilde{U}(f_1) = (5.8, 8.3, 8.7)$, $\tilde{U}(f_2) = (1.8, 6.15, 8.6)$

Comparing these fuzzy utilities by using Formula (3) we obtained: $Deg(f_1 \succ_l f_2) = 0.059$, $Deg(f_2 \succ_l f_1) = 0.023$. So, as $Deg(\tilde{f}_1 \succ_l \tilde{f}_2) > Deg(\tilde{f}_2 \succ_l \tilde{f}_1)$, then formally the best solution is $\tilde{f}_1$.

Let us solve this problem by applying CPT. As the exact objective probabilities $p(s_1), p(s_2), p(s_3)$ are not known, we need to precisiate DM's probability-related perceptions to some approximate values, like $p(s_1)=0.5, p(s_2)=0.3, p(s_3)=0.2$. Applying CPT for this case, we obtained:

$$U(f_1) = 12.4,\ \ U(f_2) = 11.35$$

Therefore, $f_1 \succ f_2$. When applying CPT, we used such numerical values for probabilities which result in the same best alternative as that obtained by the suggested approach – f_1. However, in general, it is impossible to use precise numbers as equivalents to qualitative information. In the considered problem and many real-world problems qualitative, imperfect information supported by perceptions is used to cover uncertainty as accurate precise values are not known. In turn, precise numbers and even intervals used to replace perceptions-based information are often erroneous. It is a mere guesswork disregarding a lot of useful information naturally leading to improper decisions. According to CPT, f_1 is unambiguously better than f_2. In the suggested approach, imprecision of information is better kept and takes a part in a final comparison of alternatives – there remains some vagueness on what alternative is better.

We also solved the considered problem by applying the suggested approach under precise probabilities $p(s_1)=0.2, p(s_2)=0.3, p(s_3)=0.5$ and $p(h_1)=0.7$, $p(h_2)=0.3$. The obtained results are $U(f_1)=-0.91$, $U(f_2)=19.67$. This means $f_2 \succ f_1$ and contradicts the results obtained above. So, as one can see, the use of precise probabilities in fuzzy environment may even lead to switching of preferences. The reason is that in fuzzy environment it is impossible to use reliable precise probabilities.

VIII. CONCLUSIONS

In the present study we suggested fuzzy behavioural decision model for decision making under imperfect information described in NL. In this model, decision analysis is conducted in a Cartesian space of states of nature and DM's states. A DM's state represents one principal behavior of a DM. As the latter is commonly conditioned by various mental factors, a DM's state should be in general formalized in terms of a vector-valued variable. The developed theory allows to model behaviour of a DM as depended on such mental determinants as risk attitudes, ambiguity attitudes, trust, emotions and others. The decision model is based on a fuzzy Choquet-like aggregating operator with respect to a fuzzy-valued bi-capacity. We have shown analytically and experimentally that such existing theories as Expected Utility, Choquet Expected Utility and Cumulative Prospect theory are special cases of the suggested fuzzy decision model. We applied the suggested approach to solve a benchmark and a real-life decision problem. The obtained results showed validity of the suggested approach to behavioral decision making.

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