Nonlinear Modal Reduction to Investigate the Dynamics of Disordered Systems

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***Abstract*- The study of the dynamic behaviour of disordered structures often implies the analysis of the sensitivity of the structural response to the random parameters which characterize its stiffness and damping. This consideration actually involves the solution of many more differential equations than the "ordered" structure (i.e., the structure not affected by randomness of material or geometry). In order to reduce the computational effort, a methodology is proposed, which makes use of the nonlinear normal modes to investigate both the dynamic behaviour of the structure, and its sensitivity to random parameters.**

Keywords- Random Dynamics; Disordered Structures; Nonlinear Normal Modes; Nonlinear Structures; Sensitivity Analysis

I. INTRODUCTION

Engineering design requires the computation of response quantities such as displacements, stress state, vibration frequencies against a given set of design parameters. Due to the complexity of structures or manufacture errors and inaccuracy in measurement, design parameters may be uncertain. The uncertainties of structural parameters may lead to large and unexpected excursion of the response and may lead to drastic reductions in safety. Moreover, the dynamic response of a mechanical system with uncertain parameters possesses probabilistic features which depend on the probability distribution of the system parameters. The reliability analysis for such structures strongly depends on the variation of the parameters and consequently a probabilistic approach is necessary for adequate reliability analysis.

The problem may be faced by means of Monte Carlo simulation (MCS), which allows statistical evaluation based on a large number of "deterministic" analyses with different values of the random parameters. This approach is very expensive, since it lies on the generation of a defined number of samples of the uncertain parameters and on the solution of the corresponding deterministic (linear or non-linear) problems, and is usually performed as a test for analytical approaches. In general, this kind of approach requires a strong computational effort in case of non-linear behaviour. Even if MCS is the most used among the stochastic analysis methods for structural problems, as the number of degrees of freedom (DOFs) of the structure and the number of uncertain parameters increase, the Monte Carlo structural analyses become very heavy from a computational point of view and, in some cases, the computational effort makes them inapplicable.

For this reason some non-statistical alternative procedures have been proposed in the literature. These alternative procedures mainly consist in a direct approach using probabilistic, instead of statistic, theory. This is usually pursued in both static and dynamic setting by using expansion methods, where the stiffness matrix of the structural problem is split in a deterministic part (obtained with the mean value of random parameters) and a part which accounts for the fluctuation of the random variables about its mean value.

In order to evaluate the probabilistic response, Taylor expansions or Neumann expansions ([1], [2] and [3]) are adopted to avoid inversion of matrices depending on the random parameters. These methods, in conjunction with discretization of random fields using Karhunen-Loeve expansion ([4]) or the spectral approach ([3] and [5]), allow to obtain approximate solutions in terms of moments of the response.

Most of them, as pointed out before, are based on perturbation techniques, so that the stochastic finite element (SFE) method is often identified as the classical finite element (FE) method coupled with a perturbation approach ([6] and [7]). Unfortunately, these approaches show the drawback of being less and less accurate as the level of the uncertainty of the parameters increases. Moreover, even if the uncertainty is low, they ensure accuracy only for the second order statistics of the response. Consequently, they can be applied only in the case of Gaussianity of the response, which is rarely the case, even if the basic uncertain parameters are modelled as Gaussian. In fact, due to the non-linear relationship between the system response and the basic random variables, the response is usually strongly non-Gaussian, even for linear systems. The classical SFE techniques can then be applied only to perform a second order analysis to get qualitative information on the system behaviour and for low levels of parameter uncertainties.

Particularly, a method due to Liu et al. [1] shows how to obtain an estimation of the time history of first two moments for the structural response in a linear or non-linear system. This work was improved by Chiostrini & Facchini [8] where a stochastic input has also been taken into account. The first two moments of the response at hand can be evaluated taking into account a Taylor expansion of the structural response centred on the mean value of random parameters. The method is efficient

when the dependence between the response and the random parameter is approximately a polynomial of the same degree as the expansion, but a major drawback is that it implies a heavy computational effort, as the number of equations of motion to be solved increases dramatically.

In order to reduce the number of such equations, a nonlinear normal modes approach is presented in this paper, as it has recently been examined in [16].

It is well known that the normal modes are of fundamental importance in the theory of linear dynamic conservative and non-conservative systems, as they can be used to decouple the equations governing the motion and analytically evaluate the dynamic response of the examined system.

Such an approach is clearly inapplicable in the nonlinear theory. Nevertheless, it is possible to define nonlinear normal modes (NNMs) as particular synchronous periodic solutions of the non-linear motion equations, but no link of such motions to the principle of superposition can be considered.

Several techniques can be found in specialized literature for determining the response of nonlinear systems; for free vibration problems system modes can be usefully employed to construct reduced order models: such procedures have been well developed for both linear and nonlinear systems by [13] and [14].

One such technique, introduced by [10], [11] and [12], defines the normal mode of a nonlinear oscillatory system in terms of invariant manifolds in the phase space that are tangent to the linear eigenmodes at the equilibrium point. In such a formulation, a master mode is selected, and the normal mode is constructed by a formulation in which the remaining linear modes of the system, i.e., the slave modes, depend on the master mode in a manner consistent with the system dynamics. This dependence defines the invariant manifold for the nonlinear normal mode (NNM).

The construction of the NNM invariant manifold is equivalent to the determination of the constraint relationships for all of the slave coordinates with the master coordinate; once these constraint relationships are obtained, the system dynamics can be restricted to the invariant manifold, resulting in a minimal-sized model that depends only on the master coordinates. By studying the dynamics of the reduced-order model, it is possible to recover the associated modal dynamics of the original nonlinear system.

Pesheck et al. [15] used numerical solutions of the invariant manifold equations to extend the invariant manifold approach to more general systems, including strongly nonlinear ones. In this approach, the master coordinates were expressed in polar coordinate form, and a Galerkin-based solution technique was introduced to solve the invariant manifold equations. In their work, they emphasize the fact that model reduction using NNMs can be achieved due to the important property of invariance, which is common to linear and non-linear normal modes.

II. SOME THEORETICAL REMARKS

A. Analysis of disordered systems

The governing equation of motion of a general non-linear Multiple Degrees of Freedom (MDOF) mechanical system affected by randomness in its parameters, and subject to a forcing process $\mathbf{F}(t)$, may be written in the following form:

$$\mathbf{M}\,\ddot{\mathbf{x}} + \mathbf{g}\left(\mathbf{b}, \mathbf{x}, \dot{\mathbf{x}}\right) = \mathbf{F}(t) \tag{1}$$

In the preceding equation, **M** is the mass matrix, **x** is the vector containing the structural degrees of freedom, **g** is a general (nonlinear) restoring function which can depend on the displacements **x**, the velocities $\dot{\mathbf{x}}$ and eventually on a set of random parameters, **b**. A dot over the variables denotes derivation with respect to time.

The aim is to express the dependency of the system response $\mathbf{x}(t)$ on the random vector **b** as a series expansion centred on the mean of **b** (denoted by $\mathbf{m_b}$):

$$\begin{aligned} x_h(\mathbf{b},t) \cong\; & x_h(\mathbf{m_b},t) + \frac{\partial x_h(\mathbf{m_b},t)}{\partial b_l}\left(b_l - m_{b_l}\right) + \\ & + \frac{1}{2}\frac{\partial^2 x_h(\mathbf{m_b},t)}{\partial b_l \partial b_m}\left(b_l - m_{b_l}\right)\left(b_m - m_{b_m}\right) \end{aligned} \tag{2}$$

For sake of simplicity, the following notations will be adopted from now on:

$$\bar{x}_h(t) = x_h(\mathbf{m_b},t), \qquad \bar{x}_{h,l}(t) = \frac{\partial x_h(\mathbf{m_b},t)}{\partial b_l}, \qquad \bar{x}_{h,lm}(t) = \frac{\partial^2 x_h(\mathbf{m_b},t)}{\partial b_l \partial b_m} \tag{3}$$

The first term of the RHS term in equation (2) can be computed substituting the random vector **b** with its mean $\mathbf{m_b}$ in the

equation of motion (1), thus obtaining

$$\mathbf{M}\ddot{\bar{\mathbf{x}}}+\mathbf{g}\left(\mathbf{m_b},\bar{\mathbf{x}},\dot{\bar{\mathbf{x}}}\right)=\mathbf{F}(t) \tag{4}$$

The first and second derivatives of the system response can be obtained by successively differentiating the equation of motion (1) with respect to the components of the random vector, and successively substituting the random vector **b** with its mean $\mathbf{m_b}$, thus obtaining:

$$\begin{aligned}\mathbf{M}\ddot{\bar{\mathbf{x}}}_{,l}+\mathbf{C}_T\dot{\bar{\mathbf{x}}}_{,l}+\mathbf{K}_T\bar{\mathbf{x}}_{,l}&=\mathbf{f}_1^{(l)}(t)\\ \mathbf{M}\ddot{\bar{\mathbf{x}}}_{,lm}+\mathbf{C}_T\dot{\bar{\mathbf{x}}}_{,lm}+\mathbf{K}_T\bar{\mathbf{x}}_{,lm}&=\mathbf{f}_2^{(l,m)}(t)\end{aligned} \tag{5}$$

In equation (5), the tangent damping and stiffness matrices are defined as the derivatives of the non-linear restoring function **g** with respect to the system response and velocity:

$$\mathbf{C}_T=\frac{\partial\mathbf{g}}{\partial\dot{\mathbf{x}}};\quad \mathbf{K}_T=\frac{\partial\mathbf{g}}{\partial\mathbf{x}} \tag{6}$$

As an example, if the restoring function **g** can be expressed by the product of a displacement-dependent matrix by the vector of the degrees of freedom, then the tangent damping matrix vanishes; the tangent stiffness matrix and its derivatives can be expressed as

$$\begin{aligned}&\mathbf{g}(\mathbf{x},\mathbf{b})=\mathbf{K}(\mathbf{x},\mathbf{b})\mathbf{x}\\ &K_{Tij}=\frac{\partial g_i}{\partial x_j}=\frac{\partial K_{ih}}{\partial x_j}x_h+K_{ij}\\ &\frac{\partial g_i}{\partial b_l}=\frac{\partial K_{ij}}{\partial b_l}x_j;\qquad \frac{\partial^2 g_i}{\partial b_l\partial b_m}=\frac{\partial^2 K_{ij}}{\partial b_l\partial b_m}x_j\\ &\frac{\partial K_{T\,ij}}{\partial b_l}=\frac{\partial^2 K_{ih}}{\partial b_l\partial x_j}x_h+\frac{\partial^2 K_{ih}}{\partial x_m\partial x_j}\frac{\partial x_m}{\partial b_l}x_h+\frac{\partial K_{ih}}{\partial x_j}\frac{\partial x_h}{\partial b_l}+\frac{\partial K_{ij}}{\partial b_l}+\frac{\partial K_{ij}}{\partial x_m}\frac{\partial x_m}{\partial b_l}\end{aligned} \tag{7}$$

The forcing processes $\mathbf{f}_1^{(l)}(t)$ and $\mathbf{f}_2^{(l,m)}(t)$ in equation (5) depend on the derivatives of function **g** with respect to the random parameters b_h and can be expressed as

$$\begin{aligned}\mathbf{f}_1^{(l)}(t)&=-\frac{\partial\mathbf{g}}{\partial b_l}\\ \mathbf{f}_2^{(l,m)}(t)&=-\frac{\partial\mathbf{C}_T}{\partial b_m}\dot{\bar{\mathbf{x}}}_{,l}-\frac{\partial\mathbf{K}_T}{\partial b_m}\bar{\mathbf{x}}_{,l}-\frac{\partial\mathbf{C}_T}{\partial b_l}\dot{\bar{\mathbf{x}}}_{,m}-\frac{\partial\mathbf{K}_T}{\partial b_l}\bar{\mathbf{x}}_{,m}-\frac{\partial^2\mathbf{g}}{\partial b_l\partial b_m}\end{aligned} \tag{8}$$

So far, three different systems of differential equations of motion (4) and (5) have been obtained: in the general case of a system with N degrees of freedom and affected by randomness in N_b parameters, a total number of $N\,(1+N_b+N_b^2)$ equations must be solved, which implies a heavy computational effort. Therefore, a simplifying approach is examined, namely the nonlinear normal modes.

The systems (4) and (5) can be rearranged in only one matrix expression as

$$\mathbf{M}^{(t)}\ddot{\mathbf{y}}+\mathbf{C}^{(t)}\dot{\mathbf{y}}+\mathbf{K}^{(t)}\mathbf{y}=\mathbf{f}^{(t)}(t) \tag{9}$$

Vector **y** contains all the degrees of freedom of the system, and all their first and second derivatives with respect to the random variables b_h:

$$\mathbf{y}=\left[\bar{\mathbf{x}}\ \vdots\ \cdots\ \frac{\partial\bar{\mathbf{x}}}{\partial b_h}\ \cdots\ \vdots\ \cdots\ \frac{\partial^2\bar{\mathbf{x}}}{\partial b_l\,\partial b_m}\ \cdots\right]^t \tag{10}$$

It is clear that the vector **y** can be subdivided in three blocks, the first one made by N components $\bar{x}_1\ \cdots\ \bar{x}_N$, the second by $N\cdot N_b$ derivatives $\bar{x}_{1,1}\ \cdots\ \bar{x}_{N,N_b}$ and the third one by by $N\cdot N_b^2$ second derivatives $\bar{x}_{1,11}\ \cdots\ \bar{x}_{N,N_bN_b}$. Accordingly, the mass, damping and stiffness matrices can be arranged by the corresponding blocks:

$$\mathbf{M}^{(t)} = \begin{bmatrix} \mathbf{M} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M} \end{bmatrix} \quad \mathbf{K}^{(t)} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{K}_T \end{bmatrix} \tag{11}$$

The damping matrix $\mathbf{C}^{(t)}$ is completely analogous to the stiffness matrix $\mathbf{K}^{(t)}$, with the tangent damping matrix $\mathbf{C}_T$ in place of $\mathbf{K}_T$.

Eventually, the forcing term is made up by

$$\mathbf{f}^{(t)}(t) = \left[\mathbf{F}(t) - \mathbf{g}(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{b}) \;\vdots\; \mathbf{f}_1^{(l)}(t) \;\vdots\; \mathbf{f}_2^{(l,m)}(t) \right]^t \tag{12}$$

with l and m ranging from 1 to N_b.

Once the vector $\mathbf{y}$ has been computed, the first statistical moments of the h-th DOF can be obtained as

$$\begin{aligned} m_{x_h} &\cong \bar{x}_h(t) + \frac{1}{2} \frac{\partial^2 \bar{x}_h}{\partial b_i \partial b_j} E\left[(b_i - \bar{b}_i)(b_j - \bar{b}_j) \right] \\ \sigma^2_{x_h} &\cong \frac{\partial \bar{x}_h}{\partial b_i} \frac{\partial \bar{x}_h}{\partial b_j} E\left[(b_i - \bar{b}_i)(b_j - \bar{b}_j) \right] \end{aligned} \tag{13}$$

B. Implementation of non-linear normal modes

Non-linear normal modes may be seen as a particular synchronous periodic solution of the non-linear equations of motion; although this definition can be effectively employed to establish algorithms to compute NNMs, in the present paper an equivalent definition is exploited, following [15], which builds upon the invariant manifold formulation developed by Shaw and Pierre [10], [11] and [12]. This non-linear extension of modal analysis allows to generate reduced-order models by restricting non-essential modes (or degrees of freedom) to an invariant manifold parameterized by a set of retained modes (or degrees of freedom).

It is to be kept in mind that the differential equations defined by (9) are non-linear due to following facts:

- Function **g** is a nonlinear function of displacements and velocities;
- Damping and stiffness tangent matrices are displacement and velocity-dependent.

Nevertheless, a linear part in damping and stiffness matrices can usually be identifiable and it will be denoted by $\mathbf{C}_{(L)}{}^{(t)}$ and $\mathbf{K}_L{}^{(t)}$, respectively. The equation of motion can be therefore recast as

$$\mathbf{M}^{(t)}\ddot{\mathbf{y}} + \mathbf{C}_L^{(t)}\dot{\mathbf{y}} + \mathbf{K}_L^{(t)}\mathbf{y} = \mathbf{f}^{(t)}(t) - \left[\mathbf{C}^{(t)} - \mathbf{C}_L^{(t)}\right]\dot{\mathbf{y}} - \left[\mathbf{K}^{(t)} - \mathbf{K}_L^{(t)}\right]\mathbf{y} \tag{14}$$

The case taken into examination in the present work deals with the free oscillations of an undamped Duffing oscillator; in this case, the equations of motion can be diagonalized, obtaining

$$\ddot{\boldsymbol{\eta}} + \boldsymbol{\Omega}\boldsymbol{\eta} = \mathbf{f}(\boldsymbol{\eta}) \tag{15}$$

where $\boldsymbol{\eta} = \boldsymbol{\Phi}\mathbf{y}$ is a vector of normalized modal co-ordinates, $\boldsymbol{\Phi}$ is the matrix containing the linearized modal shapes of the system (14), $\boldsymbol{\Omega}$ is the diagonal stiffness matrix whose elements are the squared circular frequencies of the linearized system, ω_k^2, $\mathbf{f}(\boldsymbol{\eta})$ contains any additional linear or non-linear term, and each overdot indicates a time derivative.

In a single-mode invariant manifold formulation [12], all degrees of freedom are constrained to be functions of a chosen co-ordinate pair, as follows, where the subscript m indicates the 'master' node of interest. If η_m denotes the master degree of freedom, its evolution in time, in a set of cylindrical coordinates, is described by the expression:

$$\eta_m = a\cos\phi \;;\; \dot{\eta}_m = -a\omega_m \sin\phi \tag{16}$$

The time history of amplitude a and phase ϕ of the master DOF must satisfy

$$\dot{a} = -\frac{f_m(\boldsymbol{\eta})}{\omega_m}\sin\phi \;;\; \dot{\phi} = \omega_m - \frac{f_m(\boldsymbol{\eta})}{a\omega_m}\cos\phi \tag{17}$$

The time histories of the remaining degrees of freedom ("slave" degrees) can be expressed by means of proper functions

$$\eta_i = P_i(a,\phi) \;;\; \dot{\eta}_i = Q_i(a,\phi) \tag{18}$$

Functions P_i and Q_i are unknown, but their dependency on the amplitude and phase of the master DOF can be modelled by means of the following differential system (see [15] and [9] for details):

$$\begin{aligned} Q_i(a,\phi) &= -\frac{\partial P_i}{\partial a}\frac{f_m(\boldsymbol{\eta})}{\omega_m}\sin\phi + \frac{\partial P_i}{\partial \phi}\left[\omega_m - \frac{f_m(\boldsymbol{\eta})}{a\omega_m}\cos\phi\right] \\ f_i(\boldsymbol{\eta}) - \omega_i^2 P_i(a,\phi) &= -\frac{\partial Q_i}{\partial a}\frac{f_m(\boldsymbol{\eta})}{\omega_m}\sin\phi + \frac{\partial Q_i}{\partial \phi}\left[\omega_m - \frac{f_m(\boldsymbol{\eta})}{a\omega_m}\cos\phi\right] \end{aligned} \tag{19}$$

The obtained equations are independent of time. Such equations describe the geometry of the considered non normal mode.

When the evaluation of the non-normal mode is completely performed, the corresponding vector **y** can be evaluated by means of

$$\begin{aligned} \mathbf{y} &= \boldsymbol{\Phi}^t\boldsymbol{\eta} \\ &= \boldsymbol{\Phi}^t\begin{bmatrix} P_1 & \cdots & P_{m-1} & a\cos\phi & P_{m+1} & & P_{N_{tot}} \end{bmatrix} \end{aligned} \tag{20}$$

The projection in the lagrangian coordinates and the resolution of the second group of equations (15) gives the desired approximation for the dependence of the system response **x** on the random parameters vector **b**.

III. APPLICATION: THE UNDAMPED DISORDERED DUFFING OSCILLATOR

As an illustrative example, the non-linear undamped multidimensional Duffing system is considered:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \lambda(\mathbf{x}^{\mathbf{t}}\mathbf{C}\mathbf{x})\mathbf{K}\mathbf{x} = \mathbf{0} \tag{21}$$

The following parameters have been chosen to characterize the oscillator: **M**=[1 0;0 2], **K**=[10 -5;-5 5], **C**=[1 -0.5;-0.5 1] ; λ is a random parameter whose mean will be taken in the following two examples λ_m=0.001 and 0.01 respectively.

The governing equations of the system response and its derivatives are

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}(1+\lambda_m\mathbf{x}^t\mathbf{C}\mathbf{x})\mathbf{x} = \mathbf{0} \tag{22}$$

$$\begin{aligned} &\mathbf{M}\ddot{\mathbf{x}}_{,\lambda} + \mathbf{K}\left[\mathbf{x}^t\mathbf{C}\mathbf{x} + \lambda_m(\mathbf{x}_{,\lambda}^t\,\mathbf{C}\mathbf{x} + \mathbf{x}^t\mathbf{C}\mathbf{x}_{,\lambda})\right]\mathbf{x} \\ &+ \mathbf{K}(1+\lambda_m\mathbf{x}^t\mathbf{C}\mathbf{x})\mathbf{x}_{,\lambda} = \mathbf{0} \end{aligned} \tag{23}$$

and

$$\begin{aligned} &\mathbf{M}\ddot{\mathbf{x}}_{,\lambda\lambda} + \left(2\mathbf{x}_{,\lambda}^t\,\mathbf{C}\mathbf{x} + 2\mathbf{x}^t\mathbf{C}\mathbf{x}_{,\lambda}\right)\mathbf{K}\mathbf{x} + \\ &+ \lambda_m\left(\mathbf{x}_{,\lambda\lambda}^t\,\mathbf{C}\mathbf{x} + 2\mathbf{x}_{,\lambda}^t\,\mathbf{C}\mathbf{x}_{,\lambda} + \mathbf{x}^t\mathbf{C}\mathbf{x}_{,\lambda\lambda}\right)\mathbf{K}\mathbf{x} + \\ &+ 2\mathbf{K}\left[\mathbf{x}^t\mathbf{C}\mathbf{x} + \lambda_m\left(\mathbf{x}_{,\lambda}^t\,\mathbf{C}\mathbf{x} + \mathbf{x}^t\mathbf{C}\mathbf{x}_{,\lambda}\right)\right]\mathbf{x}_{,\lambda} + \\ &+ \mathbf{K}\left(1+\lambda_m\mathbf{x}^t\mathbf{C}\mathbf{x}\right)\mathbf{x}_{,\lambda\lambda} = \mathbf{0} \end{aligned} \tag{24}$$

This set of equations can be solved separately; first, system (22) is solved and the associated linear system gives ω_1=1.047 rad/s, ω_2= 3.377 rad/s and $\boldsymbol{\Phi}$= [0.369 -0.929; 0.657 0.261].

The non-linear mode shapes are generated for the first group of equations while the second and third group of equations are solved directly integrating the differential equations. This is possible because the first group of equations does not depend from the derivate ones of **x**. A possible future development of the method is to solve also according to group and the third by means of use of the NNMs.

The application is made with λ_m=0.001 and results are shown in Fig. 1 and Fig. 2, while the results obtained for λ_m=0.01 in Fig. 3 and Fig. 4. The numeric simulations for the free oscillation are shown in Fig. 5 to Fig. 8 for λ_m=0.001 and for λ_m=0.01.

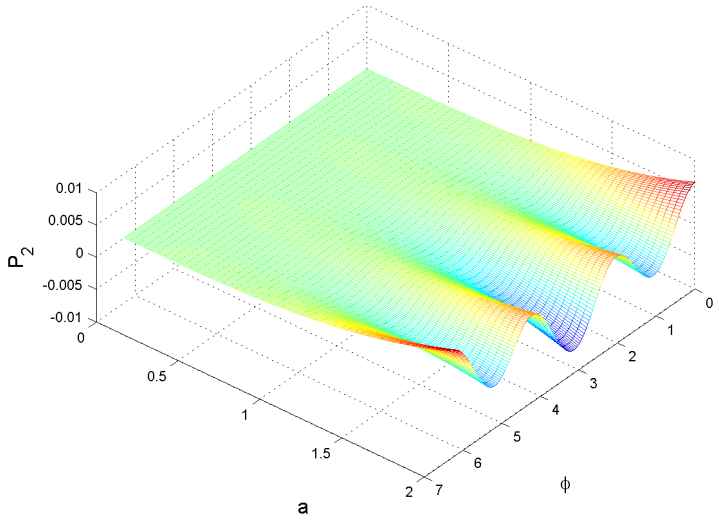

Fig. 1 Non-linear modal shape P_2 for λ_m =0.001 and N_a=2, N_ϕ=4

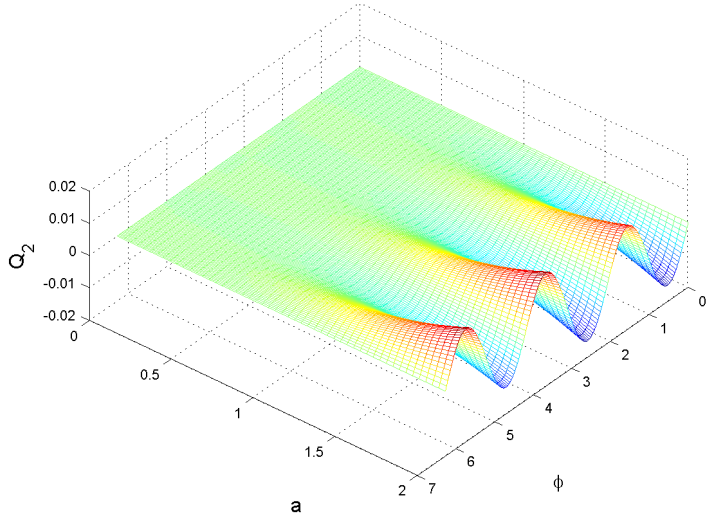

Fig. 2 Non-linear modal shape Q_2 for λ_m =0.001 and N_a=2, N_ϕ=4

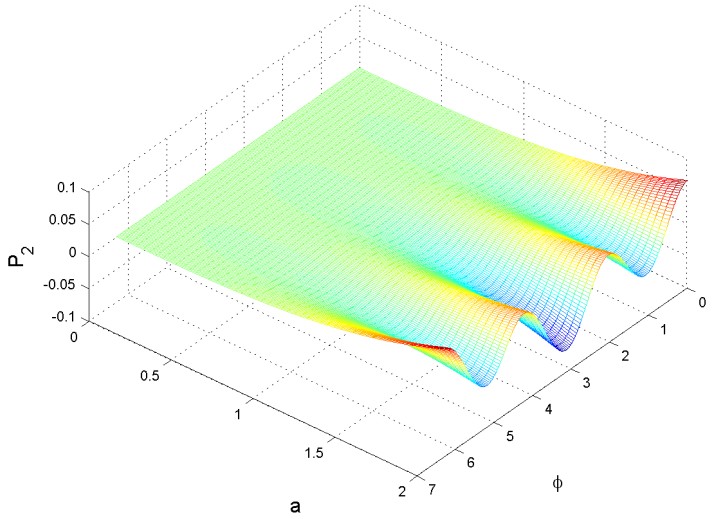

Fig. 3 Non-linear modal shape P_2 for λ_m =0.01 and N_a=2, N_ϕ=4

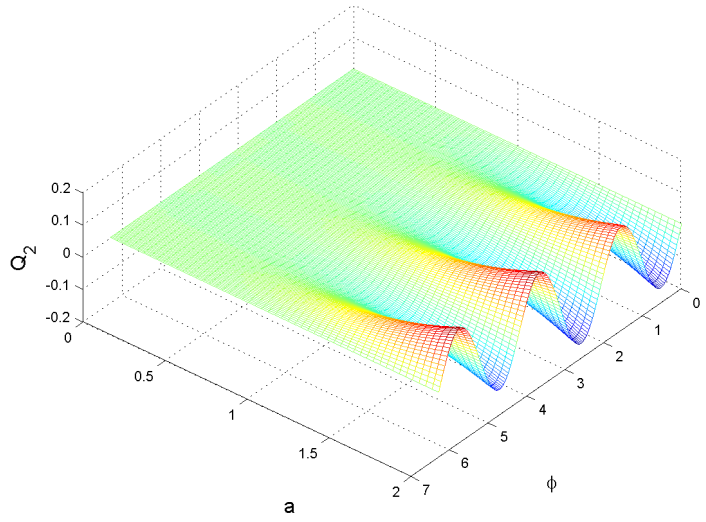

Fig. 4 Non-linear modal shape Q_2 for λ_m =0.01 and N_a=2, N_ϕ=4

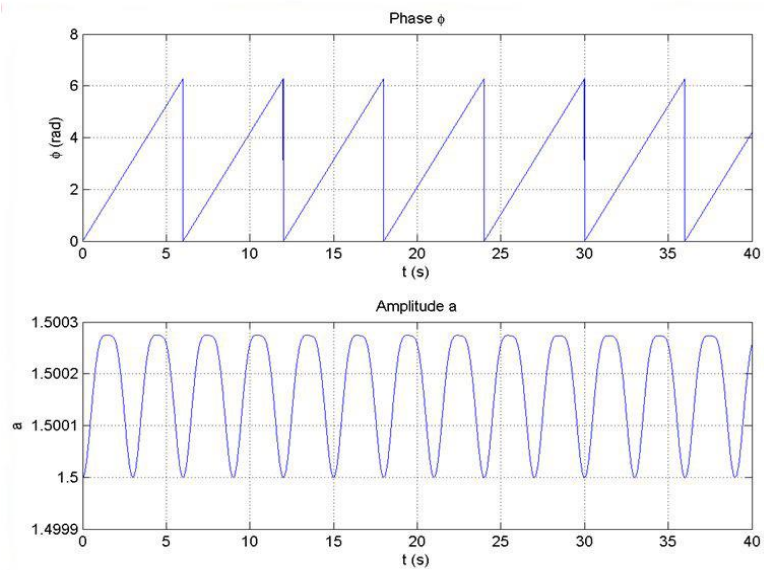

Fig. 5 Numerical simulation of phase ϕ and amplitude a obtained for λ_m =0.001

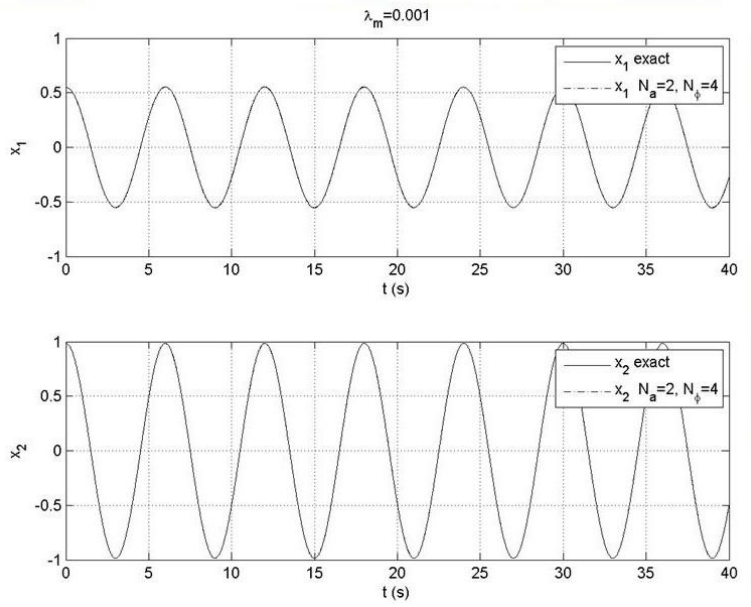

Fig. 6 Numerical simulation of the two degrees of freedom obtained for λ_m =0.001

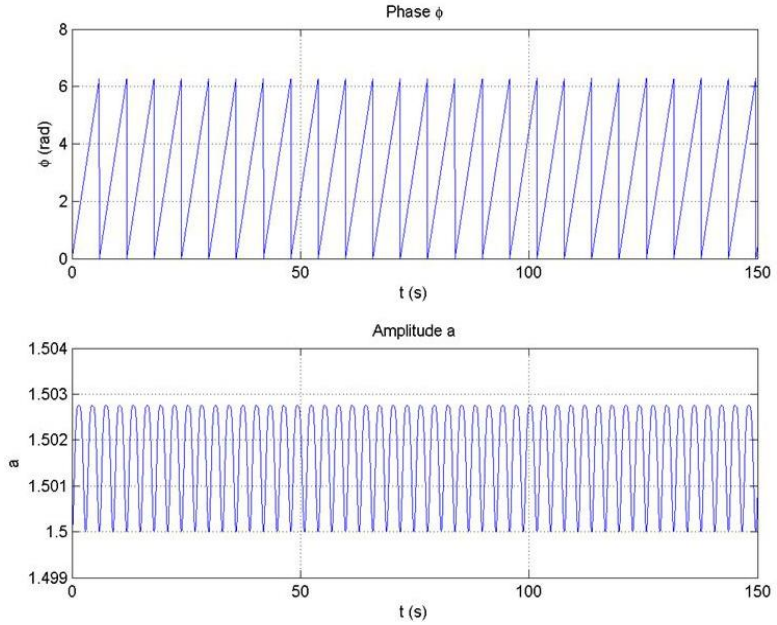

Fig. 7 Numerical simulation of phase ϕ and amplitude a obtained for λ_m =0.01

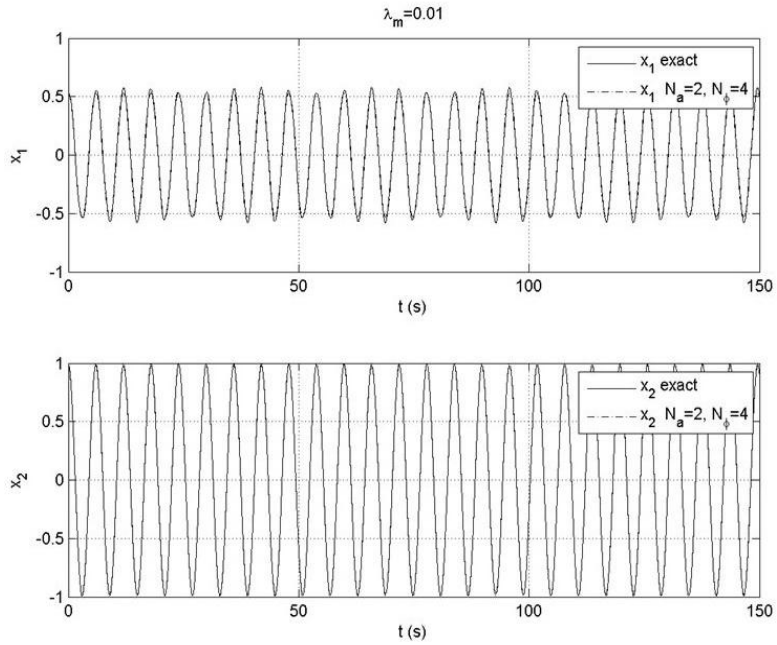

Fig. 8 Numerical simulation of the two degrees of freedom obtained for λ_m =0.01

The comparison of the NNMs method and the numerical Monte Carlo simulations, shown in Fig. 5 to Fig. 8, gives undistinguishable results. In order to compute the expected value and the standard deviation of the DOFs x_1 and x_2 the following approximations in [1] and [8] are used:

$$E[x] = x(\lambda_m) + \frac{1}{2} x_{,\lambda\lambda} \cdot \sigma_\lambda^2 \tag{25}$$

$$\sigma_x^2 \cong x,_\lambda^2 \cdot \sigma_\lambda^2 \tag{26}$$

$$x \cong x(\lambda_m) + (\lambda - \lambda_m) \cdot x,_\lambda + \frac{1}{2} x,_{\lambda\lambda} \cdot (\lambda - \lambda_m)^2 \tag{27}$$

In the following figures, the variations of x_1 and x_2 with λ and time t are reported. The coefficient of variation assumed for λ in the numerical simulations is 50%.

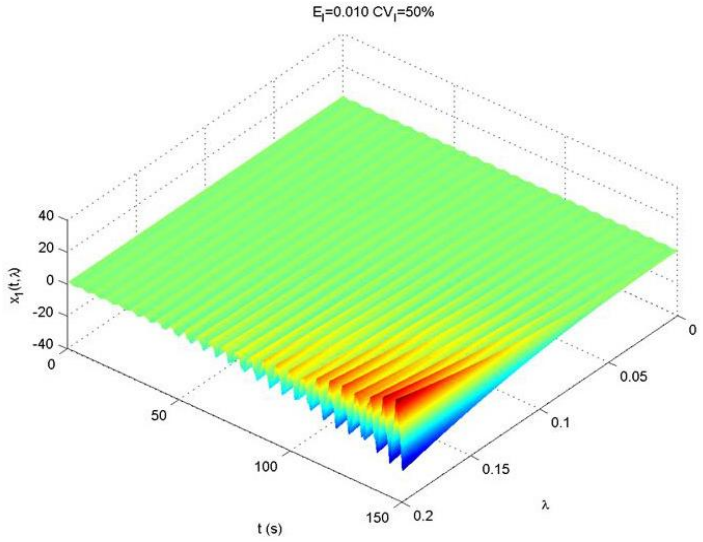

Fig. 9 Numerical simulation of the system response $x_1(t, \lambda)$ obtained with λ_m =0.01

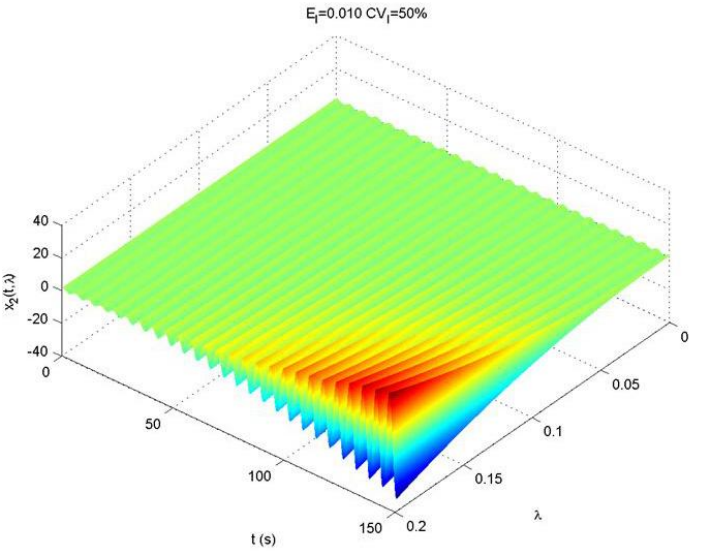

Fig. 10 Numerical simulation of the system response $x_2(t, \lambda)$ obtained with λ_m =0.01

In order to validate the method, 1000 dynamic analyses with λ_m=0.01 and coefficient of variation 50% have been simulated. In the corresponding probability distribution is shown.

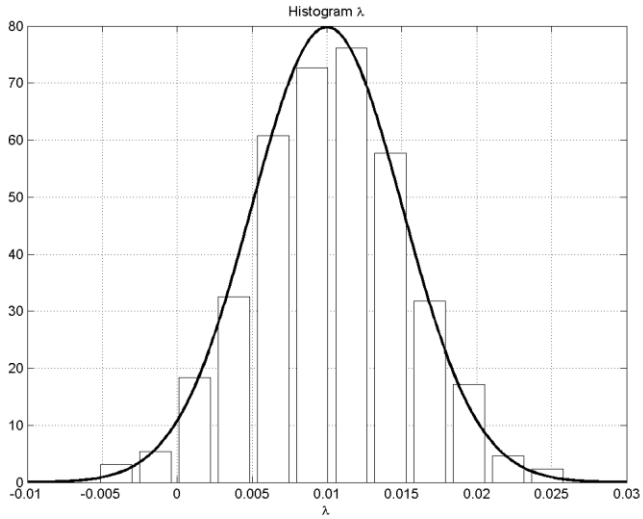

Fig. 11 Histogram and underlying normal probability distribution for the numerical simulation obtained with λ_m =0.01 and C.o.V. 50%

The comparison between the adopted method and the simulations is shown in the following figures. The NNM's method and the simulations give undistinguishable results.

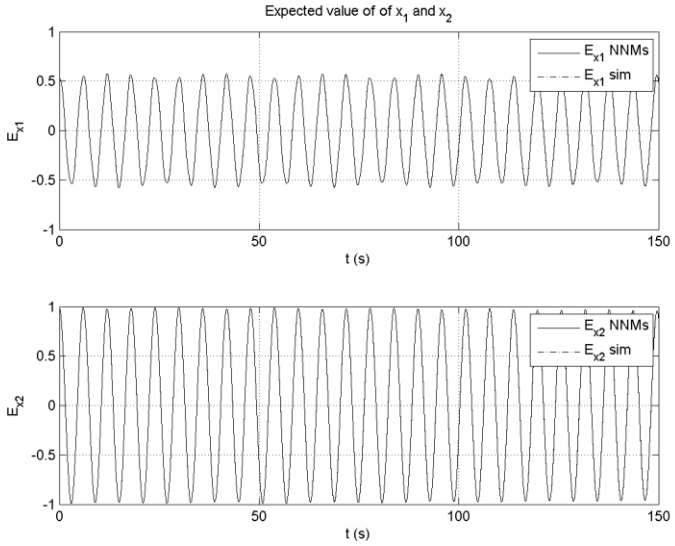

Fig. 12 Numerical simulation of the expected value system response E[x_1] and E[x_2] obtained with λ_m =0.01

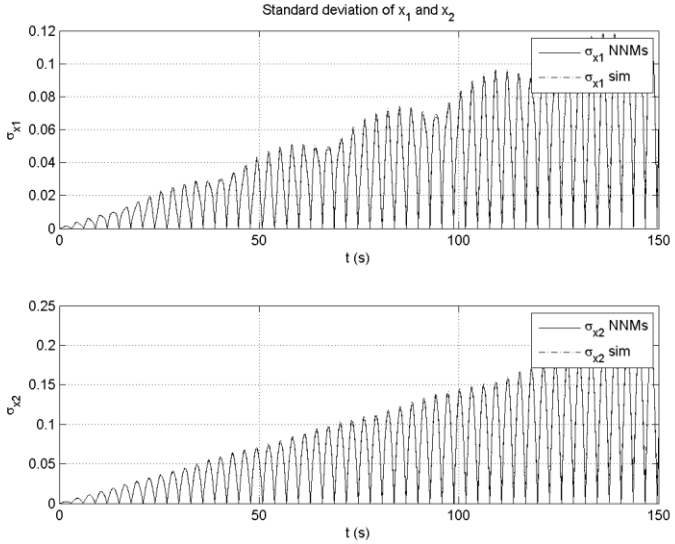

Fig. 13 Numerical simulation of the standard deviation of the system response σ_{x1} and σ_{x2} obtained with λ_m =0.01

IV. CONCLUSIONS

The study of the dynamical behaviour of disordered structures often implies a huge number of differential equations to be solved, even by means of the simplest approaches. Such drawback often makes it impossible to adequately represent the dependence of the structural response on the random parameters involved. A procedure which makes use of non-linear normal modes has been tested to reduce the dimension of the differential system to be solved. An application for a 2-DOFs duffing undamped oscillator with one random parameter has been shown.

The conventional approach would require 6 differential equations to be solved; instead, once the normal modes have been computed, only 2 differential equations are needed, thus assessing the usefulness of the proposed approach.

The comparison with Monte Carlo simulation is undistinguishable from the results given by the NNMs approach.

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