Mechanical Stability of Moderately Thick Functionally Graded Plates

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***Abstract*- In this article, an analytical approach for mechanical stability analysis of moderately thick functionally graded rectangular plates is presented. The equilibrium and stability equations are derived according to the first order shear deformation theory. Using Navier's-type solution these equations are solved for the functionally graded rectangular plate with simply supported under different types of mechanical loading. The excellent accuracy of the present analytical solution is confirmed by making some comparisons of the present results with those available in the literature. Furthermore, the effects of power of functionally graded material, plate thickness, aspect ratio, loading types on the critical buckling load of the functionally graded rectangular plate are studied and discussed in details.**

***Keywords*- Buckling; first order shear deformation theory; functionally graded materials; mechanical loads**

I. INTRODUCTION

The recent studies on new performance materials have addressed new materials known as functionally graded materials (FGMs). These are high performance, heat resistance materials able to withstand ultra high temperature and extremely large thermal gradients used in aerospace industries. Plates are one of the most important structures in engineering that are used widely in different fields. Buckling of plates especially plates made of functionally graded materials has been considered by engineers as a new field for researches, recently.

Buckling and post buckling behaviors are one of main interest in design of structural components such as plates, shells and panels for optimal and safe usage. Therefore, it is important to study the buckling and post buckling behaviors of FGM plates under mechanical, thermal and combined thermomechanical loads for accurate and reliable design. Some works about the stability of FGM structures relating to present study are introduced in the following.

Javaheri and Eslami [1-3] and Shariat and Eslami [4] reported mechanical and thermal buckling of rectangular functionally graded plates by using the classical plate theory [1, 2] and higher order shear deformation plate theory [3, 4]. They used energy method to derive governing equations that analytically solved to obtain the closed-form solutions of critical loading. The same authors and Shariat [5-7] extended their studies when influences of initial geometrical imperfection on the critical buckling loading are taken into consideration. Bouazza et al [8] studied the buckling response of simply supported of Sigmoid FGM plate using the first order shear deformation theory .The thermal buckling behaviours under uniform, linear and sinusoidal temperature rise across the thickness. Bouazza et al. [9] used the first-order shear deformation theory to derive closed-form relations for buckling temperature difference of simply supported moderately thick rectangular power-law (linear, quadratic, cubic, and inverse quadratic) functionally graded plates. Bouazza et al [10] presented the derivation of equations for mechanical buckling of rectangular thin functionally graded plates under uniaxial and biaxial compression using classic plate theory. Buckling analysis of isotropic rectangular plates on elastic foundation was carried out by Yu [11]. He considered a plate resting on elastic foundation with two opposite edges simply supported and different boundary conditions along the other edges (Levy solution). It was concluded that the number of waves of the buckling mode increases with stiffness and occurs in the direction of the applied stress. Also, increase in the plate width or height increases the buckling load for the free horizontal edge case, but the effect is the opposite for the clamped or simply supported cases. Meisam et al. [12] investigated the buckling behavior of functionally graded material plate under different loading conditions based on the classical plate theory (Levy solution); the governing equations are obtained for functionally graded rectangular plates using the principle of minimum total potential energy. The response of a functionally graded ceramic–metal plate was investigated by Praveen and Reddy using a finite element model that accounts for the transverse shear strains, rotary inertia, and moderately large rotations in the Von Karman sense [13].

In the present article, the equilibrium and stability equations for FGM are obtained on the basis of first-order shear deformation theory and Navier’s solution. Resulting equations are employed to obtain the closed-form solutions for the critical buckling load. The results are compared with the results of previous works in the literature.

II. ANALYSIS

A. Functionally graded materials

Consider a case when FGM plate made up of a mixture of ceramic and metal as show in Fig1. The material properties vary continuously across the thickness according to the following, which are the same as the equations proposed by Praveen and Reddy. [13]

$$\begin{aligned} &E(z)=E_m+E_{cm}V_f(z) && E_{cm}=E_c-E_m \\ &\alpha(z)=\alpha_m+\alpha_{cm}V_f(z) && \alpha_{cm}=\alpha_c-\alpha_m \\ &\nu(z)=\nu_0 \end{aligned} \tag{1}$$

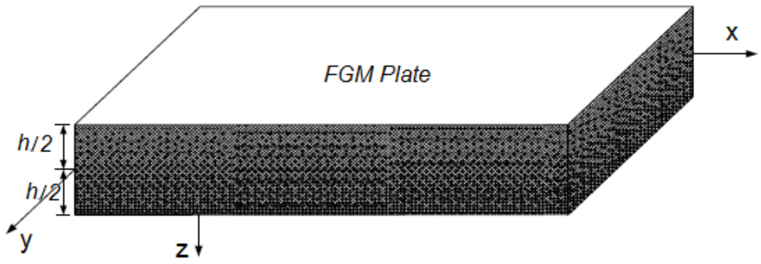

Fig. 1 Typical FGM rectangular plate

where subscripts m and c refer to properties of metal and ceramic, respectively, and $V_f(z)$ is volume fraction of the constituents which can be mostly defined by power–law functions [8,14-15]. For power-law FGM, volume fraction function is expressed as

$$V_f(z)=\left(z/h+1/2\right)^k \tag{2}$$

B. Stability equations

Assume that u,v,w denote the displacements of the neutral plane of the plate in x, y, z directions respectively; ϕ_x , ϕ_y denote the rotations of the normals to the plate midplane. According to the first- order shear deformation theory, the strains of the plate can be expressed as [8-9, 16]

$$\begin{aligned} &\varepsilon_x=u_{,x}+z\phi_{x,x} \qquad \varepsilon_y=\upsilon_{,y}+z\phi_{y,y} \\ &\gamma_{xy}=u_{,y}+\upsilon_{,x}+z(\phi_{x,y}+\phi_{y,x}) \\ &\gamma_{xz}=\phi_x+w_{,x} \qquad \gamma_{zy}=\phi_y+w_{,y} \end{aligned} \tag{3}$$

The forces and moments per unit length of the plate expressed in terms of the stress components through the thickness are

$$N_{ij}=\int_{-h/2}^{h/2}\sigma_{ij}\,dz \;\; ; \;\; M_{ij}=\int_{-h/2}^{h/2}\sigma_{ij}z\,dz \quad ; \quad Q_{ij}=\int_{-h/2}^{h/2}\tau_{ij}\,dz \tag{4}$$

The nonlinear equations of equilibrium according to Von Karman’s theory are given by

$$\begin{aligned} &N_{x,xx}+2N_{xy,xy}+N_{y,yy}=0 \\ &M_{x,xx}+2M_{xy,xy}+M_{y,yy}-Q_{x,x}-Q_{y,y}=0 \\ &Q_{x,x}+Q_{y,y}+q+N_x w_{,xx}+N_y w_{,yy}+2N_{xy}w_{,xy}=0 \end{aligned} \tag{5}$$

Using Eqs. (1) (3) and (4), the equilibrium Eq. (5) may be reduced to a set of one equation as

$$\begin{aligned} &\nabla^4 w+\frac{2(1+\nu)}{E_1}\nabla^2(N_x w_{,xx}+N_y w_{,yy}+2N_{xy}w_{,xy}+q) \\ &\qquad -\frac{E_1(1-\nu^2)}{E_1E_3-E_2^2}(N_x w_{,xx}+N_y w_{,yy}+2N_{xy}w_{,xy}+q)=0 \end{aligned} \tag{6}$$

where

$$(E_1, E_2, E_3) = \int_{-h/2}^{h/2} (1, z, z^2)\, E(z) dz \tag{7}$$

To establish the stability equations, the critical equilibrium method is used. Assuming that the state of stable equilibrium of a general plate under mechanical load may be designated by w_0. The displacement of the neighboring state is $w_0 + w_1$, where w_1 is an arbitrarily small increment of displacement. Substituting $w_0 + w_1$ into Eq. (6) and subtracting the original equation, results in the following stability equation

$$\begin{aligned}&\nabla^4 w_1 + \frac{2(1+\nu)}{E_1} \nabla^2 (N_x^0 w_{1,xx} + N_y^0 w_{1,yy} + 2N_{xy}^0 w_{1,xy}) \\ &- \frac{E_1(1-\nu^2)}{E_1E_3 - E_2^2} (N_x^0 w_{1,xx} + N_y^0 w_{1,yy} + 2N_{xy}^0 w_{1,xy}) = 0\end{aligned} \tag{8}$$

where, N_x^0, N_y^0 *and* N_{xy}^0 refer to the pre-buckling force resultants.

The simply supported boundary condition is defined as

$$\begin{aligned} w_1 = 0\,,\ M_{x1} = 0\,,\ \phi_{y1} = 0 \ \ on\ \ x = 0, a \\ w_1 = 0\,,\ M_{y1} = 0\,,\ \phi_{x1} = 0 \ \ on\ \ y = 0, b \end{aligned} \tag{9}$$

The following approximate solution is seen to satisfy both the governing equation and the boundary conditions

$$w_1 = c \sin(m\pi x / a) \sin(n\pi y / b) \tag{10}$$

where *m*, *n* are number of half waves in the *x* and *y* directions, respectively, and *c* is a constant coefficient.

C. Mechanical buckling analysis

Consider a rectangular plate with the length a and width b which is subjected to in-plane loads as shown in Fig. 2. Therefore, the pre-buckling forces can be obtained using the equilibrium conditions as [10, 12]

$$N_x^0 = \xi_1 P_1\ ,\ \ N_y^0 = \xi_2 P_1\ ,\ \ N_{xy}^0 = 0 \tag{11}$$

where P_1 is the force per unit length, ξ_1 and ξ_2 are the load parameter which indicate the loading conditions. Negative values for ξ_1 and ξ_2 indicate that plate is subjected to biaxial compressive loads. Also, zero value for ξ_1 or ξ_2 shows uniaxial loading in x or y directions, respectively. Substituting Eq (11) into Eq (8), one obtains

$$\nabla^4 w_1 + \frac{2(1+\nu)}{E_1} P_1\, \nabla^2 (\xi_1 w_{1,xx} + \xi_2 w_{1,yy}) - \frac{E_1(1-\nu^2)}{E_1E_3 - E_2^2} P_1\, (\xi_1 w_{1,xx} + \xi_2 w_{1,yy}) = 0 \tag{12}$$

Substituting Eq. (10) into Eq. (12), and substituting for the buckling load P_1

$$P_1 = \frac{(E_1E_3 - E_2^2)\pi^2 (m^2 + n^2 B_a^2)^2 E_1}{-2(1+\nu)(E_1E_3 - E_2^2)\pi^2 (\xi_1 m^4 + (\xi_1 + \xi_2) m^2 n^2 B_a^2 + \xi_2 n^4 B_a^4) - E_1^2 a^2 (1-\nu^2)(\xi_1 m^2 + \xi_2 n^2 B_a^2)}$$

The critical buckling load, P_{1cr}, is the smallest value of P_1 which is obtained when *m* = 1 and *n* = 1. Therefore,

$$P_{1cr} = \frac{(E_1E_3 - E_2^2)\pi^2 (1 + B_a^2)^2 E_1}{-2(1+\nu)(E_1E_3 - E_2^2)\pi^2 (\xi_1 + (\xi_1 + \xi_2) B_a^2 + \xi_2 B_a^4) - E_1^2 a^2 (1-\nu^2)(\xi_1 + \xi_2 B_a^2)} \tag{13}$$

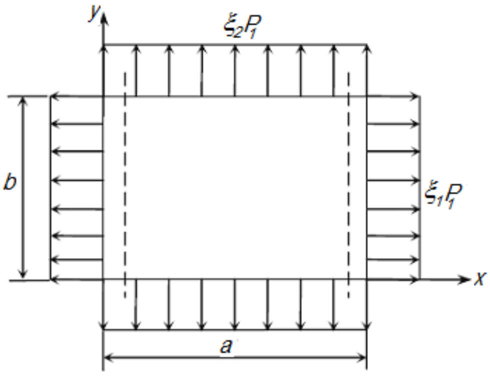

Fig. 2 The rectangular plate subjected to in plane loads

III. NUMERICAL COMPUTATIONS AND RESULTS

A. Validation of the Results

Based on the derived formulation, a computer program is developed to study the behavior of FGM plates in mechanical buckling. The analysis is performed for pure materials and different values of volume fraction exponent, k, for aluminium–alumina FGM. The Young's modulus and Poisson's ratio for aluminium are: 70 GPa and 0.3 and for alumina: 380GPa and 0.3, respectively. Note that the Poisson's ratio is chosen to be 0.3 for simplicity.

In order to validate the results, a comparison with the previously known works has been carried out. In Table 1, the non-dimensional critical buckling load is presented in order to compare with Refs. [10], [11] and [12] for an isotropic plate (k=0) with different aspect ratios. As the table shows, the present results have a good agreement with Refs. [10], [11] and [12].

TABLE 1 COMPARISON OF THE NON-DIMENSIONAL CRITICAL BUCKLING LOAD $(P_{1cr} \cdot a^2 / D)$ FOR AN ISOTROPIC PLATE (K=0), (B=1, H=0.01).

a/b	(ξ_1, ξ_2)	Ref[11]	Ref[12]	Ref[10]	Present study
0.5	(-1,0)	15.42	15.4212	15.4213	15.4032
	(-1,-1)	12.33	12.3370	12.3370	12.3225
1	(-1,0)	39.23	39.4784	39.4784	39.4599
	(-1,-1)	19.74	19.7392	19.7392	19.7299

To validate the analytical analysis formulation, the critical buckling temperatures of simply supported FGM plates are compared with results of Bouazza [10] et al andMeisam el al [12] in Table 2. Both results are in excellent agreement.

TABLE 2 COMPARISON OF THE CRITICAL BUCKLING LOAD (MN/M) FOR A FGM PLATE (B=1, H=0.01).

K	a/b	Critical buckling load								
		(ξ_1, ξ_2) =(-1,0)			(ξ_1, ξ_2) =(0,-1)			(ξ_1, ξ_2) =(-1,-1)		
		Ref[12]	Ref[10]	Present study	Ref[12]	Ref[10]	Present study	Ref[12]	Ref[10]	Present study
0	0.5	2.14655	2.14655	2.14403	8.58619	8.58619	8.57612	1.71724	1.71724	1.71522
	1	1.37379	1.37379	1.37315	1.37379	1.37379	1.37315	0.68689	0.68690	0.68657
	1.5	1.49066	1.61230	1.61175	0.71658	0.71658	0.71633	0.49609	0.49609	0.49592
1	0.5	1.06993	1.06993	1.06887	4.27971	4.27971	4.27548	0.85594	0.85594	0.85510
	1	0.68475	0.68475	0.68448	0.68475	0.684753	0.68448	0.34238	0.34238	0.34224
	1.5	0.74300	0.80363	0.80340	0.35717	0.35717	0.35707	0.24727	0.24727	0.24720
2	0.5	0.83488	0.83488	0.83405	3.33953	3.33953	3.33619	0.66791	0.66791	0.66724
	1	0.53432	0.53433	0.53411	0.53432	0.53433	0.53411	0.26716	0.26716	0.26706
	1.5	0.57978	0.62709	0.62690	0.27871	0.27871	0.27862	0.19295	0.19295	0.19290

B. Results

Fig.3 indicates the typical examples of the ceramic volume fraction along the volume fraction index. Fig. 4 shows the variation trend of critical buckling load with respect to the plate aspect ratio a/b for different values of material gradient index k. The relative thickness of the plate is set as h/a= 0.2, for a simply supported plate subjected to biaxial compression. It is observed that with increasing the plate aspect ratio a/b from 1 to 10, the critical buckling load also increases steadily, whatever the material gradient index k is.

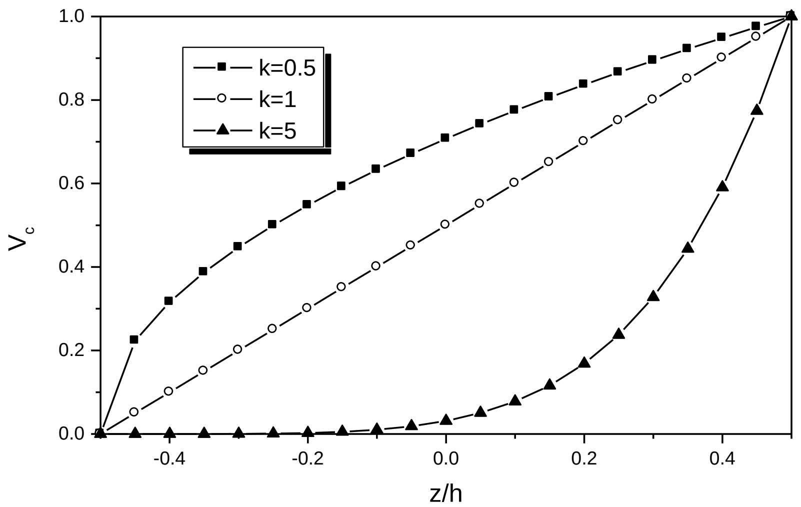

Fig. 3 Volume fraction of ceramic along the thickness direction

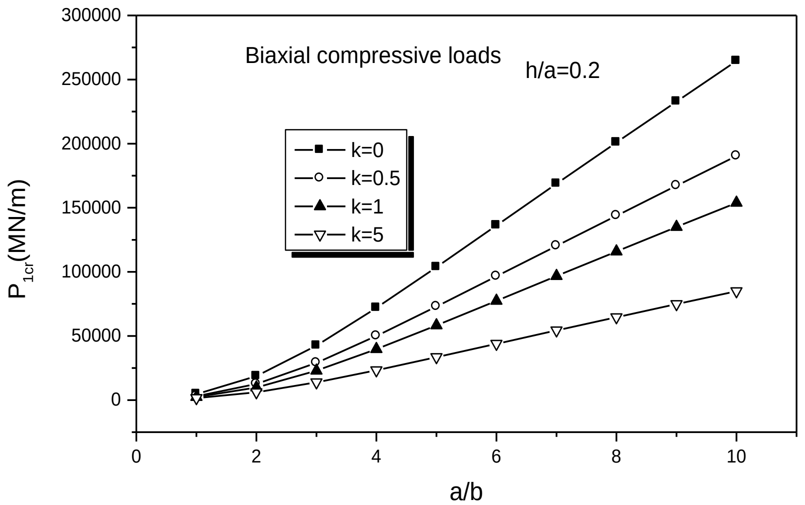

Fig. 4 Critical buckling load for a simply supported plate versus the aspect ratio of the plate, under biaxial compression

The critical buckling load versus the thickness to side ratio (h/a) for a FGM plate with simply supported subjected to biaxial compression with different values of power of FGM, is shown in Fig. 5. It can be found that increasing the thickness of the plate severely increases the critical buckling load, whatever the material gradient index k is. Such behavior is due to the influence of the transverse shear deformation in the plate.

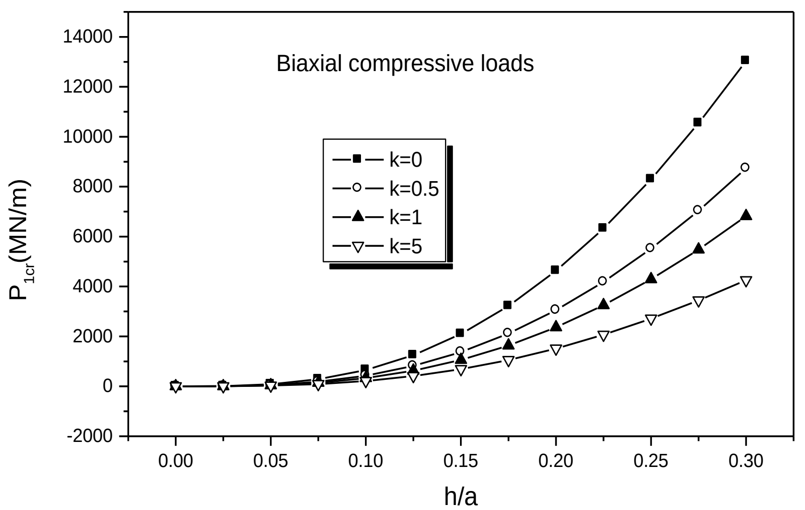

Fig. 5 Critical buckling load for a simply supported plate versus the thickness to side ratio, under biaxial compression

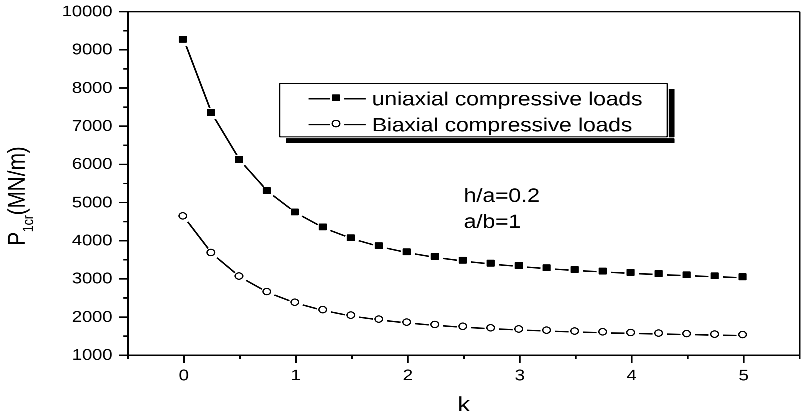

Fig. 6 Critical buckling load for a plate versus the power of FGM

Fig. 6 illustrates the variation of the critical buckling load of square FGM versus the power of FGM, under different types of in-plane mechanical loading based on the first order shear deformation theory. The thickness -to-side ratio h/a is assumed to be 0.2. It is shown the critical buckling load decreases as the power of FGM increases. On the other hand, the buckling load of the plate under uniaxial compression is greater than the one under biaxial compression.

IV. CONCLUSIONS

The buckling analyses of fully simply supported rectangular FGM plates under mechanical are investigated by the first order shear deformation theory. The mechanical loadings are assumed to be; uniaxial compression, and biaxial compression. Based on the numerical results, the following conclusions are reached:

1. In the case of mechanical loads, the critical buckling mode varies with respect to the load parameter ξ_1 and ξ_2 or the aspect ratio a/b.

2. The buckling load of the plate under uniaxial compression is greater than the one under biaxial compression.

3. Volume fraction index k is increased, the critical buckling load decreases. This is because as volume fraction index is increased, the contained quantity of ceramic decreases.

4. Geometric parameter h/a is increased, the critical buckling load increases rapidly.

REFERENCES

[1] Javaheri R, Eslami MR. *Buckling of functionally graded plates under in-plane compressive loading*. ZAMM 2002; 82(4):277–83.

[2] Javaheri R, Eslami MR. *Thermal buckling of functionally graded plates*. AIAA J 2002; 40(1):162–9.

[3] Javaheri R, Eslami MR. *Thermal buckling of functionally graded plates based on higher order theory*. J Therm Stress 2002; 25(1):603–25.

[4] Samsam Shariat BA, Eslami MR. *Buckling of thick functionally graded plates under mechanical and thermal loads*. Compos Struct 2007; 78:433–9.

[5] Samsam Shariat BA, Javaheri R, Eslami MR. *Buckling of imperfect functionally graded plates under in-plane compressive loading*. Thin-Wall Struct 2005; 43:1020–36.

[6] Samsam Shariat BA, Eslami MR. *Thermal buckling of imperfect functionally graded plates*. Int J Solids Struct 2006; 43: 4082–96.

[7] Samsam Shariat BA, Eslami MR. *Effect of initial imperfection on thermal buckling of functionally graded plates*. J Therm Stress 2005; 28: 1183–98.

[8] M. Bouazza, A. Tounsi, E.A. Adda-Bedia, A. Megueni, *Buckling analysis of functionally graded plates with simply supported edges*. Leonardo Journal of Sciences (15) (2009) 21–32, ISSN: 1583-0233.

[9] M. Bouazza , A. Tounsi , E. A. Adda-Bedia, A . Megueni. *Thermoelastic stability analysis of functionally graded plates: An analytical approach*. Elsevier, Computational Materials Science 49 (2010) pp. 865-870, ISSN 0927-0256.

[10] M. Bouazza, D. Ouinas, Y. Abdelaziz and A. Hamouine. *Buckling of thin plates under uniaxial and biaxial compression.* Journal of Materials Science and Engineering B 2 (8) (2012) 487-492.

[11] Yu, L.H, *Buckling of rectangular plates on an elastic foundation using the Levy solution*. AIAA. 46, (2008)

[12] Meisam Mohammadi, Ali Reza Saidi ,Emad Jomehzadeh . *Levy Solution for Buckling Analysis of Functionally Graded Rectangular Plates*. Int J Appl Compos Mater, DOI 10.1007/s10443-009-9100-z, Springer Science + Business Media B.V. 2009.

[13] Praveen GN, Reddy JN. *Nonlinear transient thermoelastic analysis of functionally graded ceramic-metal plates*. International Journal of Solids and Structures 1998; 35: 4457-4476.

[14] Chi SH, Chung YL. *Mechanical behavior of functionally graded material plates under transverse load-Part I: Analysis.* Int J Solids Struct 2006; 43: 3657–74.

[15] Chi SH, Chung YL. *Mechanical behavior of functionally graded material plates under transverse load- Part II: Numerical results.* Int J Solids Struct 2006; 43: 3675–91.

[16] Leissa AW. *Vibration of plates*. NASA, SP-160, (1969).