# Generalized Thermoelastic Diffusion With Effect of Fractional Parameter on Plane Waves TemperatureDependent Elastic Medium 

Sarhan. Y. Atwa<br>Faculty of Science, Department of Mathematics, Zagazig University, P.O. Box 44519, Zagazig, Egypt<br>Higher Institute of Engineering, Dep. of Eng. Math and Physics, Shorouk Academy, Egypt srhan_1@ yahoo.com


#### Abstract

The present paper is concerned with the investigation of disturbances in a homogeneous thermoelastic diffusion. The formulation is applied to the generali thermoelasticity, isotropic temperature-dependent elastic medium with fractional order eneralized based on the fractional time derivatives under the effect of diffusion. The analytical expressions for displacement components, stresses, temperature field, concentration and chemical potential are obtained in the physical domain by using the normal mode analysis technique. These expressions are calculated numerically for a copper-like material and shown graphically. Effect of fractional parameter and presence of diffusion are analyzed theoretically and numerically. Comparisons are made with the results predicted by the fractional and without fractional order in the presence and absence of diffusion.


Keywords- Generalized Thermoelasticity; Fractional Parameter; Diffusion; Relaxation Time; Normal Mode Analysis

## I. INTRODUCTION

Recently, some interesting models have been proposed successfully by applying the fractional calculus to study the physical processes particularly in the area of mechanics of solids, control theory, electricity, heat conduction, diffusion problems and viscoelasticity etc. It has been verified/examined that the use of fractional order derivatives/integrals leads to the formulation of certain physical problems which is more economical and useful than the classical approach. There are some materials (e.g., porous materials, man-made and biological materials/polymers and colloids, glassy etc.) and physical situations (like low-temperature, amorphous media and transient loading etc.) where the CTE theory based on the classical Fourier's law is unsuitable. In such cases it is better to use a generalized thermoelasticity (and more generally thermo-viscoelasticity) theory based on an anomalous heat conduction theory involving fractional time-derivatives, see Ignaczak (2010). Abel is the first author, who applied fractional calculus to obtain the solution of an integral equation arising in the formulation of the tautochrone problem. After Abel's study, great attention has been devoted to the major study of fractional calculus by Liouville. Fractional order derivatives have been employed for the description of viscoelastic materials by Caputo and Mainardi (1971a; 1971b) and (Caputo 1974) and they have established the connection between fractional derivatives and the linear theory of viscoelasticity. They also obtained a very good agreement with the experimental results successfully. In (Rabotnov (1966), Mainardi (1998)) one can find many applications of fractional calculus to various problems of mechanics of solids. A considerable research effort has been extended to study anomalous diffusion that is characterized by the time-fractional diffusion wave equation introduced by Kimmich Kimmich (2002).

During recent years, fractional calculus has also been introduced in the field of thermoelasticity. Povstenko (2005) has constructed a quasi-static uncoupled thermoelasticity model based on the heat conduction equation with a fractional order time derivative. He used the Caputo (1967) fractional derivative and obtained the stress components corresponding to the fundamental solution of a Cauchy problem for the fractional order heat conduction equation in both the one-dimensional and two-dimensional cases. Ezzat and Fayik (2011) constructed a model in generalized thermoelastic diffusion by using fractional time-derivatives.

Diffusion can be defined as the movement of particles from an area of high concentration to an area of lower concentration until equilibrium is reached. It occurs as a result of the second law of thermodynamics which states that the entropy or disorder of any system must always increase with time. Diffusion is important in many life processes. There is now a great deal of interest in the study of this phenomenon, due to its many applications in geophysics and industrial applications. In integrated circuit fabrication, diffusion is used to introduce dopants in controlled amounts into the semiconductor substrate. In particular, diffusion is used to form the base and emitter in bipolar transistors, form integrated resistors, form the source/drain regions in MOS transistors and dope poly-silicon gates in MOS transistors. In most of these applications, the concentration is calculated using what is known as Fick's law. This is a simple law that does not take into consideration the mutual interaction between the introduced substance and the medium into which it is introduced or the effect of the temperature on this interaction. The phenomenon of diffusion is used to improve the conditions of oil extractions (seeking ways of more efficiently recovering oil from oil deposits). These days, oil companies are interested in the process of thermoelastic diffusion for more efficient
extraction of oil from oil deposits. The thermodiffusion process also helps the investigation in the field associated with the advent of semiconductor devices and the advancement of microelectronics.

Thermodiffusion in the solids is one of the transport processes that have great practical importance. Most of the research associated with the presence of concentration and temperature gradients has been made with metals and alloys. The first critical review was published in the work of Oriani (1969). With the advancement of a nuclear energetic the interest in thermodiffusion has returned to metallic oxides that often heats up in an inhomogeneous temperature field Fryxel and Aitken (1969) in connection with technological conditions.

Thermodiffusion in an elastic solid is due to the coupling of the fields of temperature, mass diffusion and that of strain. Heat and mass exchange with the environment during the process of thermodiffusion in an elastic solid. The concept of thermodiffusion is used to describe the process of thermomechanical treatment of metals (carbonizing, nitriding steel, etc.), these processes are thermally activated, and their diffusing substances being, e.g, nitrogen, carbon etc. They are accompanied by deformations of the solid. Nowacki (1974a; 1974b; 1974c; 1976) developed the theory of thermoelastic diffusion. In this theory, the coupled thermoelastic model is used. This implies infinite speeds of propagation of thermoelastic waves. Sherief et al. (2004) developed the theory of generalized thermoelastic diffusion that predicts finite speeds of propagation for thermoelastic and diffusive waves. The reflection phenomena of P and SV waves from the free surface of an elastic solid with thermodiffusion was considered by Singh (2005). Sherief and Saleh (2005) worked on a problem of a thermoelastic half-space with a permeating substance in contact with the bounding plane in the context of the theory of generalized thermoelastic diffusion with one relaxation time. Recently,

Thomas (1980) theory was based on Fundamentals of Heat Transfer, and Othman et al. (2009) studied the effect of diffusion on the two-dimensional problem of generalized thermoelasticity with Green and Naghdi theory. Owing to the mathematical difficulties encountered in two-dimensional multi-field coupled generalized heat conduction problems, the problems become too complicated to obtain an analytical solution. Instead of analytical methods, several authors applied numerical techniques such as finite difference method, finite element method, boundary value method etc. for solving such kind of problems. In recent years, normal mode analysis method has been applied to study various problem of generalized thermoelasticity Othman et al (2005; 2008a 2008b; 2012a; 2011; 2012b; 2012c).

The present study is motivated by the importance of thermoelastic diffusion process in the field of oil extraction. The theory of thermodiffusion is also applied in the description of thermo-mechanical treatment of porous media of sintered powder metals. Thermodiffusion methods have been successfully applied in the last few years in improving the mechanical properties of product made of powder metals.

The present paper is concerned with the investigation of disturbances in a homogeneous, isotropic temperature-dependent elastic medium with fractional order generalized thermo-diffusion. The formulation is applied to the generalized thermoelasticity based on the fractional time derivatives under the effect of diffusion. The analytical expressions for displacement components, stresses, temperature field, concentration and chemical potential are obtained in the physical domain by using the normal mode analysis techniques. These expressions are calculated numerically for a copper-like material and depicted graphically. Effect of fractional parameter and presence of diffusion are analyzed theoretically and numerically. Comparisons are made with the results predicted by the fractional and without fractional order in the presence and absence of diffusion.

## II. FORMULATION OF THE PROBLEM

Let us consider an isotropic, homogeneous, thermally and perfectly conducting elastic medium with temperature-dependent of modulus of elasticity. We consider an orthogonal Cartesian coordinate system oxyz having originated on the surface $\mathrm{z}=0$ and oz being a line vertically downward.

Following Ezzat and Fayik (2011) the governing equations for an isotropic, homogeneous temperature dependent elastic solid with generalized thermo-diffusion at uniform temperature $T_{0}$ in the undisturbed state, in the absence of external body forces and heat sources are:
(1) the equation of motion

$$
\begin{equation*}
\rho \ddot{\mathrm{u}}=\sigma_{\mathrm{ij}, \mathrm{j}} \tag{1}
\end{equation*}
$$

Where $\rho$ is the density, $\mathbf{u}$ is the displacement vector, $\sigma_{\mathrm{ij}}$ are the components of the stress tensor
(2) The strain-displacement relation

$$
\begin{equation*}
\mathrm{e}_{\mathrm{ij}}=\frac{1}{2}\left(\mathrm{u}_{\mathrm{i}, \mathrm{j}}+\mathrm{u}_{\mathrm{j}, \mathrm{i}}\right), \tag{2}
\end{equation*}
$$

(3) the constitutive equations

$$
\begin{gather*}
\sigma_{i j}=2 \mu \mathrm{e}_{\mathrm{ij}}+\left[\lambda \mathrm{e}_{\mathrm{kk}}-v\left(\mathrm{~T}-\mathrm{T}_{0}\right)-\beta C\right] \delta_{\mathrm{ij}}  \tag{3}\\
\mathrm{P}=-\beta \mathrm{e}_{\mathrm{kk}}+\mathrm{bC}-\mathrm{a}\left(\mathrm{~T}-\mathrm{T}_{0}\right) \tag{4}
\end{gather*}
$$

Where $\mathrm{e}_{\mathrm{ij}}$ are the components of strain tensor, T is the absolute temperature, C is the concentration of the diffusive material in the elastic medium, $\lambda, \mu$ are Lame's constant, $v$ and $\beta$ are the material constants given by $v=(3 \lambda+2 \mu) \alpha_{t}$ and $\beta=(3 \lambda+2 \mu) \alpha_{c}, \alpha_{t}$ is the coefficient of linear thermal expansion, $\alpha_{c}$ is the coefficient of linear diffusion expansion, $P$ is the chemical potential, $a$ is the measure of thermodiffusion effect and $b$ is the measure of diffusive effect.
(4) the energy equation with fractional order time derivatives Ezzat and Fayik (2011)

$$
\begin{equation*}
\mathrm{K} \nabla^{2} \mathrm{~T}=\frac{\partial}{\partial \mathrm{t}}\left(1+\frac{\tau_{0}^{\alpha}}{\alpha!} \frac{\partial^{\alpha}}{\partial \mathrm{t}^{\alpha}}\right)\left(\rho \mathrm{C}_{\mathrm{E}} \mathrm{~T}+v \mathrm{~T}_{\mathrm{o}} \mathrm{e}+\mathrm{a}_{\mathrm{o}} \mathrm{C}\right), \quad 0 \prec \alpha \leq 1, \tag{5}
\end{equation*}
$$

Where $K$ is the thermal conductivity, $\mathrm{C}_{\mathrm{E}}$ is the specific heat at constant strain, $\mathrm{T}_{\mathrm{o}}$ is the temperature of the medium in its natural state assumed to be such that $\left|\frac{T-T_{0}}{T_{0}}\right| \ll 1$, e is the cubical dilatation given by $e=\nabla . u$ is the thermal relaxation time and

$$
\frac{\partial^{\alpha}}{\partial \mathrm{t}}{ }^{\alpha} f(x, t)= \begin{cases}f(x, t)-f(x, 0) & \text { when } \alpha \rightarrow 0, \\ I^{1-\alpha} \frac{\partial f(x, t)}{\partial \mathrm{t}} & \text { when } 0 \prec \alpha \prec 1, \\ \frac{\partial f(x, t)}{\partial \mathrm{t}} & \text { when } \alpha=1 .\end{cases}
$$

In the above definition, the Riemann-Liouville fractional integral operator $I^{\alpha}$ is defined as

$$
I^{\alpha} f(t)=\frac{1}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1} f(s) d s
$$

Where $\Gamma(\alpha)$ is the well-known Gamma function.
(5) the generalized diffusion equation

$$
\begin{equation*}
{\mathrm{d} \beta \mathrm{e}_{\mathrm{kk}, \mathrm{ii}}+\mathrm{daT}_{, \mathrm{ii}}+\frac{\partial}{\partial \mathrm{t}}\left(1+\frac{\tau_{0}^{\alpha}}{\alpha!} \frac{\partial^{\alpha}}{\partial \mathrm{t}^{\alpha}}\right) \mathrm{C}-\mathrm{dbC}_{, \mathrm{ii}}=0, \quad 0 \prec \alpha \leq 1, ~}_{\text {, }} \tag{6}
\end{equation*}
$$

where d is the diffusion coefficient and $\tau$ is the diffusion relaxation time. Also note that in the above equations, a comma followed by a suffix denotes material derivative and a superposed dot denotes the derivative with respect to time $t$. We consider all quantities are functions of the coordinates $\mathrm{x}, \mathrm{z}$ and t . The displacement components thus have the following form

$$
\begin{equation*}
\mathrm{u}_{\mathrm{x}}=\mathrm{u}(\mathrm{x}, \mathrm{z}, \mathrm{t}), \quad \mathrm{u}_{\mathrm{y}}=0, \quad \mathrm{u}_{\mathrm{z}}=\mathrm{w}(\mathrm{x}, \mathrm{z}, \mathrm{t}) \tag{7}
\end{equation*}
$$

Now, we assume that

$$
\begin{equation*}
\lambda=\lambda_{0}\left(1-\alpha^{*} T_{0}\right), \quad \mu=\mu_{0}\left(1-\alpha^{*} T_{0}\right), \quad v=v_{0}\left(1-\alpha^{*} T_{0}\right), \quad \beta=\beta_{0}\left(1-\alpha^{*} T_{0}\right), \tag{8}
\end{equation*}
$$

Where $\lambda_{0}, \mu_{0}, v_{0}$ and $\beta_{0}$ are constants and $\alpha^{*}$ is the linear temperature coefficient.
In the case of the modulus elasticity is temperature independent, $\alpha^{*}=0$.
By substituting from Eq. (8) in Eqs. (3) and (4), we obtain

$$
\begin{gather*}
\alpha_{1} \sigma_{x x}=\left(\lambda_{0}+2 \mu_{0}\right) \mathrm{e}_{\mathrm{xx}}+\lambda_{0} \mathrm{e}_{\mathrm{zz}}-v_{0}\left(\mathrm{~T}-\mathrm{T}_{0}\right)-\beta_{0} \mathrm{C},  \tag{9}\\
\alpha_{1} \sigma_{\mathrm{zz}}=\left(\lambda_{0}+2 \mu_{0}\right) \mathrm{e}_{\mathrm{zz}}+\lambda_{0} \mathrm{e}_{\mathrm{xx}}-v_{0}\left(\mathrm{~T}-\mathrm{T}_{0}\right)-\beta_{0} \mathrm{C},  \tag{10}\\
\alpha_{1} \sigma_{\mathrm{xz}}=2 \mu_{0} \mathrm{e}_{\mathrm{xz}},  \tag{11}\\
\mathrm{P}=-\frac{\beta_{0}}{\alpha_{1}} \mathrm{e}_{\mathrm{kk}}+\mathrm{bC}-\mathrm{a}\left(\mathrm{~T}-\mathrm{T}_{0}\right), \tag{12}
\end{gather*}
$$

Where

$$
\begin{equation*}
\alpha_{1}=\frac{1}{\left(1-\alpha^{*} T_{0}\right)} \tag{13}
\end{equation*}
$$

By using Eqs. (9)-(11) in Eq. (1), we get

$$
\begin{align*}
& \alpha_{1} \rho \ddot{\mathrm{u}}=\left(\lambda_{0}+\mu_{0}\right) \mathrm{e}_{, \mathrm{x}}+\mu_{0} \nabla^{2} \mathrm{u}-v_{0} \mathrm{~T}_{, \mathrm{x}}-\beta_{0} \mathrm{C}_{, \mathrm{x}},  \tag{14}\\
& \alpha_{1} \rho \ddot{\mathrm{w}}=\left(\lambda_{0}+\mu_{0}\right) \mathrm{e}_{, \mathrm{z}}+\mu_{0} \nabla^{2} w-v_{0} \mathrm{~T}_{, \mathrm{z}}-\beta_{0} \mathrm{C}_{, \mathrm{z}} . \tag{15}
\end{align*}
$$

By substituting from Eq. (8) in Eqs. (5) and (6), we obtain

$$
\begin{align*}
& \mathrm{K} \nabla^{2} \mathrm{~T}=\frac{\partial}{\partial \mathrm{t}}\left(1+\frac{\tau_{0}^{\alpha}}{\alpha!} \frac{\partial^{\alpha}}{\partial \mathrm{t}^{\alpha}}\right)\left(\rho \mathrm{C}_{\mathrm{E}} \mathrm{~T}+\frac{v_{0} \mathrm{~T}_{\mathrm{o}}}{\alpha_{1}} \mathrm{e}+\mathrm{aT}_{\mathrm{o}} \mathrm{C}\right),  \tag{16}\\
& \frac{\mathrm{d} \beta}{\alpha_{1}} \nabla^{2} \mathrm{e}+\mathrm{da} \nabla^{2} \mathrm{~T}+\frac{\partial}{\partial \mathrm{t}}\left(1+\frac{\tau_{0}^{\alpha}}{\alpha!} \frac{\partial^{\alpha}}{\partial \mathrm{t}^{\alpha}}\right) \mathrm{C}-\mathrm{db} \nabla^{2} \mathrm{C}=0, \tag{17}
\end{align*}
$$

To transform the above equations in non-dimensional forms, we will use the following non-dimensional variables

$$
\begin{gathered}
\left(\mathrm{x}^{\prime}, \mathrm{z}^{\prime}\right)=\frac{\tilde{\omega}}{\mathrm{c}_{1}}(\mathrm{x}, \mathrm{z}), \quad\left(\mathrm{u}^{\prime}, \mathrm{w} \mathrm{w}^{\prime}\right)=\frac{\tilde{\omega}}{\mathrm{c}_{1}}(\mathrm{u}, \mathrm{w}), \quad\left(\mathrm{t}^{\prime}, \tau_{0}^{\prime}, \tau^{\prime}\right)=\tilde{\omega}\left(\mathrm{t}, \tau_{0}, \tau\right), \\
\sigma_{\mathrm{ij}}^{\prime}=\frac{\sigma_{\mathrm{ij}}}{\rho \mathrm{c}_{1}^{2}}, \mathrm{C}^{\prime}=\frac{\beta_{0}}{\rho \mathrm{c}_{1}^{2}} \mathrm{C}, \theta^{\prime}=\frac{v_{0}\left(\mathrm{~T}-\mathrm{T}_{0}\right)}{\rho c_{1}^{2}}, \mathrm{P}^{\prime}=\frac{\mathrm{P}}{\beta_{0}}
\end{gathered}
$$

Where

$$
\tilde{\omega}=\frac{\rho \mathrm{C}_{\mathrm{E}} \mathrm{c}_{1}^{2}}{\mathrm{~K}} \text { and } \mathrm{c}_{1}^{2}=\frac{\lambda_{0}+2 \mu_{0}}{\rho}
$$

Using these non-dimensional variables, equations take the following forms (omitting the primes for convenience)

$$
\begin{gather*}
\ddot{\mathrm{u}}=\frac{1}{\alpha_{1}}\left[\beta_{1} \mathrm{e}_{, \mathrm{x}}+\left(1-\beta_{1}\right) \nabla^{2} \mathrm{u}-\theta_{, \mathrm{x}}-\mathrm{C}_{, \mathrm{x}}\right],  \tag{18}\\
\ddot{\mathrm{w}}=\frac{1}{\alpha_{1}}\left[\beta_{1} \mathrm{e}_{, \mathrm{z}}+\left(1-\beta_{1}\right) \nabla^{2} \mathrm{w}-\theta_{, \mathrm{z}}-\mathrm{C}_{, \mathrm{z}}\right],  \tag{19}\\
\mathrm{K} \nabla^{2} \theta=\frac{\partial}{\partial \mathrm{t}}\left(1+\frac{\tau_{0}^{\alpha}}{\alpha!} \frac{\partial^{\alpha}}{\partial \mathrm{t}^{\alpha}}\right)\left(\theta+\frac{\delta \delta_{0}}{\alpha_{1}} \mathrm{e}+\mathrm{a}_{1} \delta_{0} \mathrm{C}\right),  \tag{20}\\
\nabla^{2} \mathrm{e}+\alpha_{1} \alpha_{2} \nabla^{2} \theta+\alpha_{1} \alpha_{3} \frac{\partial}{\partial \mathrm{t}}\left(1+\frac{\tau_{0}^{\alpha}}{\alpha!} \frac{\partial^{\alpha}}{\partial \mathrm{t}}{ }^{\alpha}\right) \mathrm{C}-\alpha_{1} \alpha_{4} \nabla^{2} \mathrm{C}=0, \tag{21}
\end{gather*}
$$

$$
\begin{gather*}
\sigma_{\mathrm{xx}}=\frac{1}{\alpha_{1}}\left[\mathrm{u}_{, \mathrm{x}}+\left(2 \beta_{1}-1\right) \mathrm{w}_{, \mathrm{z}}-\theta-\mathrm{C}\right],  \tag{22}\\
\sigma_{\mathrm{zz}}=\frac{1}{\alpha_{1}}\left[\mathrm{w}_{, \mathrm{z}}+\left(2 \beta_{1}-1\right) \mathrm{u}_{, \mathrm{x}}-\theta-\mathrm{C}\right],  \tag{23}\\
\sigma_{\mathrm{xz}}=\frac{1-\beta_{1}}{\alpha_{1}}\left[\mathrm{u}_{, \mathrm{z}}+\mathrm{w}_{, \mathrm{x}}\right],  \tag{24}\\
\mathrm{P}=-\frac{1}{\alpha_{1}}\left(\mathrm{u}_{, \mathrm{x}}+\mathrm{w}_{, \mathrm{z}}\right)+\alpha_{4} \mathrm{C}-\alpha_{2} \theta . \tag{25}
\end{gather*}
$$

Where

$$
\delta=\frac{v_{0}}{\rho \mathrm{C}_{\mathrm{E}}}, \delta_{0}=\frac{v_{0} \mathrm{~T}_{0}}{\rho \mathrm{c}_{1}^{2}}, \mathrm{a}_{1}=\frac{\mathrm{ac}_{1}^{2}}{\beta_{0} \mathrm{C}_{\mathrm{E}}}, \alpha_{2}=\frac{\mathrm{a} \mathrm{\rho c}_{1}^{2}}{\beta_{0} v_{0}}, \quad \alpha_{3}=\frac{\mathrm{Kc}_{1}^{2}}{\mathrm{~d} \beta_{0}^{2} \mathrm{C}_{\mathrm{E}}}, \quad \alpha_{4}=\frac{\mathrm{bpc}_{1}^{2}}{\beta_{0}^{2}}, \beta_{1}=\frac{\left(\lambda_{0}+\mu_{0}\right)}{\rho c_{1}^{2}}
$$

Introducing the potential functions $\varphi(\mathrm{x}, \mathrm{z}, \mathrm{t})$ and $\psi(\mathrm{x}, \mathrm{z}, \mathrm{t})$ defined by the relations in the non-dimensional form:

$$
\begin{equation*}
\mathrm{u}=\left(\varphi_{, \mathrm{x}}-\psi_{, \mathrm{z}}\right), \quad \mathrm{w}=\left(\varphi_{, \mathrm{z}}+\psi_{, \mathrm{x}}\right) \tag{26}
\end{equation*}
$$

By substituting Eq. (26) in Eqs. (18)-(21), we get

$$
\begin{gather*}
{\left[\nabla^{2}-\alpha_{1} \frac{\partial^{2}}{\partial \mathrm{t}^{2}}\right] \varphi=\theta+\mathrm{C}}  \tag{27}\\
{\left[\left(1-\beta_{1}\right) \nabla^{2}-\alpha_{1} \frac{\partial^{2}}{\partial \mathrm{t}^{2}}\right] \psi=0,}  \tag{28}\\
{\left[\nabla^{2}-\frac{\partial}{\partial \mathrm{t}}\left(1+\frac{\tau_{0}^{\alpha}}{\alpha!} \frac{\partial^{\alpha}}{\partial \mathrm{t}}{ }^{\alpha}\right)\right] \theta-\delta_{0} \frac{\partial}{\partial \mathrm{t}}\left(1+\frac{\tau_{0}^{\alpha}}{\alpha!} \frac{\partial^{\alpha}}{\partial \mathrm{t}}{ }^{\alpha}\right)\left(\frac{\delta}{\alpha_{1}} \nabla^{2} \varphi+\mathrm{a}_{1} \mathrm{C}\right)=0,}  \tag{29}\\
\nabla^{4} \varphi+\alpha_{1} \alpha_{2} \nabla^{2} \theta+\left[\alpha_{1} \alpha_{3} \frac{\partial}{\partial \mathrm{t}}\left(1+\frac{\tau_{0}^{\alpha}}{\alpha!} \frac{\partial^{\alpha}}{\partial \mathrm{t}}{ }^{\alpha}\right)-\alpha_{1} \alpha_{4} \nabla^{2}\right] \mathrm{C}=0, \tag{30}
\end{gather*}
$$

## III. SOLUTION OF THE PROBLEM

The solution of the physical quantities can be decomposed in terms of normal modes in the following form:

$$
\begin{equation*}
\left[C, P, u, w, e, \varphi, \psi, \theta, \sigma_{i j}\right](x, z, t)=\left[C^{*}, \mathrm{P}^{*}, \mathrm{u}^{*}, \mathrm{w}^{*}, \mathrm{e}^{*}, \varphi^{*}, \psi^{*}, \theta^{*}, \sigma_{\mathrm{ij}}^{*}\right](\mathrm{z}) \exp (\omega \mathrm{t}+\mathrm{imx}) \tag{31}
\end{equation*}
$$

Where $\left[\mathrm{C}^{*}, \mathrm{P}^{*}, \mathrm{u}^{*}, \mathrm{w}^{*}, \mathrm{e}^{*}, \varphi^{*}, \Psi^{*}, \theta^{*}, \sigma_{\mathrm{ij}}^{*}\right](\mathrm{z})$ i is the imaginary unit, $\omega$ (complex) is the frequency and m is the wave number in the x -direction.

Using Eq. (31), then Eqs. (27)-(30) take the following forms

$$
\begin{gather*}
{\left[\mathrm{D}^{2}-\mathrm{m}^{2}-\alpha_{1} \omega^{2}\right] \varphi^{*}(\mathrm{z})-\theta^{*}(\mathrm{z})-\mathrm{C}^{*}(\mathrm{z})=0}  \tag{32}\\
{\left[\left(1-\beta_{1}\right)\left(\mathrm{D}^{2}-\mathrm{m}^{2}\right)-\alpha_{1} \omega^{2}\right] \psi^{*}(\mathrm{z})=0}  \tag{33}\\
\frac{\delta_{0} \delta \omega_{1}}{\alpha_{1}}\left(\mathrm{D}^{2}-\mathrm{m}^{2}\right) \varphi^{*}(\mathrm{z})-\left[\mathrm{D}^{2}-\mathrm{m}^{2}-\omega_{1}\right] \theta^{*}(\mathrm{z})+\mathrm{a}_{1} \delta_{0} \omega_{1} \mathrm{C}^{*}(\mathrm{z})=0  \tag{34}\\
\left(\mathrm{D}^{2}-\mathrm{m}^{2}\right)^{2} \varphi^{*}(\mathrm{z})+\alpha_{1} \alpha_{2}\left(\mathrm{D}^{2}-\mathrm{m}^{2}\right) \theta^{*}(\mathrm{z})+\left[\alpha_{1} \alpha_{3} \omega_{1}-\alpha_{1} \alpha_{4}\left(\mathrm{D}^{2}-\mathrm{m}^{2}\right)\right] \mathrm{C}^{*}(\mathrm{z})=0 \tag{35}
\end{gather*}
$$

Where $\mathrm{D}=\frac{\mathrm{d}}{\mathrm{dz}}, \omega_{1}=\left(\omega+\frac{\omega^{\alpha+1} \tau_{0}^{\alpha}}{\alpha!}\right)$
Eliminating $\theta^{*}(\mathrm{z})$ and $\mathrm{C}^{*}(\mathrm{z})$ between Eqs. (32), (34) and (35), after some simple computations we get the following sixth-order ordinary differential equation satisfied by $\varphi^{*}(\mathrm{z})$

$$
\begin{equation*}
\left[\mathrm{D}^{6}-1_{1} \mathrm{D}^{4}+1_{2} \mathrm{D}^{2}-1_{3}\right] \varphi^{*}(\mathrm{z})=0 . \tag{36}
\end{equation*}
$$

Where

$$
l_{1}=\frac{-1}{\alpha_{1} g_{7}}\left[g_{1} g_{5}+\alpha_{1} g_{8}+g_{4} g_{7}\right], l_{2}=\frac{1}{\alpha_{1} g_{7}}\left[g_{1} g_{6}+g_{3} g_{5}+\alpha_{1} g_{9}+g_{4} g_{8}\right], l_{3}=\frac{-1}{\alpha_{1} g_{7}}\left[g_{3} g_{6}+g_{4} g_{9}\right],
$$

And

$$
\begin{gathered}
\mathrm{g}_{1}=\delta_{0} \delta \omega_{1}+\alpha_{1} \mathrm{a}_{1} \delta_{0} \omega_{1}, \mathrm{~g}_{2}=\mathrm{a}_{1} \delta_{0} \omega_{1} \alpha_{1}^{2} \omega^{2}, \mathrm{~g}_{3}=-\mathrm{g}_{1} \mathrm{~m}^{2}-\mathrm{g}_{2}, \mathrm{~g}_{4}=\alpha_{1}\left(\mathrm{a}_{1} \delta_{0} \omega_{1}-\mathrm{m}^{2}-\omega_{1}\right), \\
\mathrm{g}_{5}=\alpha_{1}\left(\alpha_{2}+\alpha_{4}\right), \mathrm{g}_{6}=-\alpha_{1}\left(\alpha_{2}+\alpha_{4}\right) \mathrm{m}^{2}-\alpha_{1} \alpha_{3} \omega_{1}, \mathrm{~g}_{7}=1-\alpha_{1} \alpha_{4}, \\
\mathrm{~g}_{8}=-2 \mathrm{~g}_{7} \mathrm{~m}^{2}+\alpha_{1} \alpha_{3} \omega_{1}+\alpha_{1}^{2} \alpha_{4} \omega^{2}, \mathrm{~g}_{9}=\mathrm{g}_{7} \mathrm{~m}^{4}-\alpha_{1} \alpha_{3} \omega_{1} \mathrm{~m}^{2}-\alpha_{1}^{2} \alpha_{3} \omega_{1} \omega^{2}-\alpha_{1}^{2} \alpha_{4} \omega^{2} \mathrm{~m}^{2},
\end{gathered}
$$

In a similar manner, we can show that $\theta^{*}(z)$ and $C^{*}(z)$ satisfy the following equations

$$
\begin{equation*}
\left[\mathrm{D}^{6}-1_{1} \mathrm{D}^{4}+1_{2} \mathrm{D}^{2}-1_{3}\right]\left\{\varphi^{*}(\mathrm{z}), \theta^{*}(\mathrm{z}), \mathrm{C}^{*}(\mathrm{z})\right\}=0 . \tag{37}
\end{equation*}
$$

The general solution of Eq. (37) which is regular at $\mathrm{z} \rightarrow \infty$ can be written as

$$
\begin{equation*}
\varphi^{*}(\mathrm{z})=\sum_{\mathrm{j}=1}^{3} \mathrm{R}_{\mathrm{j}}(\mathrm{~m}, \omega) \mathrm{e}^{-\mathrm{k}_{\mathrm{j}} \mathrm{z}} \quad j=1,2,3, \tag{38}
\end{equation*}
$$

Where $\mathrm{k}_{\mathrm{j}}(\mathrm{j}=1,2,3)$ are the eigenvalues (roots) of the following characteristics equations

$$
\begin{equation*}
\mathrm{k}^{6}-\mathrm{l}_{1} \mathrm{k}^{4}+\mathrm{l}_{2} \mathrm{k}^{2}-\mathrm{l}_{3}=0, \tag{39}
\end{equation*}
$$

Given by

$$
\begin{gathered}
\mathrm{k}_{1}^{2}=\frac{1}{3}\left(2 \mathrm{p} \operatorname{sinq}+\mathrm{l}_{1}\right), \\
\mathrm{k}_{2}^{2}=\frac{-1}{3}\left(\mathrm{p}[\sqrt{3} \cos \mathrm{q}+\sin \mathrm{q}]-\mathrm{l}_{1}\right), \\
\mathrm{k}_{3}^{2}=\frac{1}{3}\left(\mathrm{p}[\sqrt{3} \cos \mathrm{q}-\sin \mathrm{q}]+\mathrm{l}_{1}\right),
\end{gathered}
$$

And

$$
\mathrm{p}=\sqrt{1_{1}^{2}-31_{2}}, \mathrm{q}=\frac{\sin ^{-1} \mathrm{r}}{3}, r=\frac{91_{1} 1_{2}-2 l_{1}^{3}-271_{3}}{2 \mathrm{p}^{3}} .
$$

Following the same process, we obtain the solution for $\theta^{*}(z)$ and $C^{*}(z)$ as follows

$$
\begin{equation*}
\theta^{*}(z)=\sum_{j=1}^{3} R_{j}^{\prime}(m, \omega) e^{-k_{j} z} \quad j=1,2,3 \tag{40}
\end{equation*}
$$

$$
\begin{equation*}
C^{*}(z)=\sum_{j=1}^{3} R_{j}^{\prime \prime}(m, \omega) e^{-k_{j} z} \quad j=1,2,3 \tag{41}
\end{equation*}
$$

Where $R_{j}(m, \omega), R_{j}^{\prime}(m, \omega)$ and $R_{j}^{\prime \prime}(m, \omega)$, are some parameters depending on $m$ and $\omega$ to be determined by the boundary conditions of the problem. Substituting Eqs. (38), (40) and (41) into the Eqs. (32), (34) and (35), we can easily obtain

$$
\begin{array}{ll}
R_{j}^{\prime}(m, \omega)=M_{1 j} R_{j}(m, \omega), & j=1,2,3 \\
R_{j}^{\prime \prime}(m, \omega)=M_{2 j} R_{j}(m, \omega), & j=1,2,3 \tag{43}
\end{array}
$$

We thus have

$$
\begin{align*}
& \theta^{*}(z)=\sum_{j=1}^{3} M_{1 j} R_{j}(m, \omega) e^{-k_{j} z},  \tag{44}\\
& C^{*}(z)=\sum_{j=1}^{3} M_{2 j} R_{j}(m, \omega) e^{-k_{j} z}, \tag{45}
\end{align*}
$$

Where

$$
\begin{gather*}
\mathrm{M}_{1 \mathrm{j}}=\frac{g_{1} \mathrm{k}_{\mathrm{j}}^{2}+\mathrm{g}_{3}}{\alpha_{1} \mathrm{k}_{\mathrm{j}}^{2}+\mathrm{g}_{4}},  \tag{46}\\
\mathrm{M}_{2 \mathrm{j}}=\mathrm{k}_{\mathrm{j}}^{2}-\mathrm{m}^{2}-\alpha_{1} \omega^{2}-\mathrm{M}_{1 \mathrm{j}} \tag{47}
\end{gather*}
$$

The solution of Eq. (33) can be written as

$$
\begin{equation*}
\psi^{*}(\mathrm{z})=\mathrm{R}_{4}(\mathrm{~m}, \omega) \mathrm{e}^{-\mathrm{k}_{4} z} \tag{48}
\end{equation*}
$$

Where

$$
\mathrm{k}_{4}=\sqrt{\mathrm{m}^{2}+\frac{\alpha_{1} \omega}{1-\beta_{1}}}
$$

In order to obtain the displacement components $u$ and w, using Eq. (31), Eq. (26), becomes

$$
\begin{align*}
& \mathrm{u}^{*}(\mathrm{z})=\mathrm{im} \varphi^{*}(\mathrm{z})-\mathrm{D} \psi^{*}(\mathrm{z})  \tag{49}\\
& \mathrm{w}^{*}(\mathrm{z})=\mathrm{D} \varphi^{*}(\mathrm{z})+\mathrm{im} \psi^{*}(\mathrm{z}) \tag{50}
\end{align*}
$$

Which give on using Eqs. (38) and (48)

$$
\begin{align*}
& u^{*}(z)=k_{4} R_{4}(m, \omega) e^{-k_{4} z}+i m \sum_{j=1}^{3} R_{j}(m, \omega) e^{-k_{j} z},  \tag{51}\\
& w^{*}(z)=i m R_{4}(m, \omega) e^{-k_{4} z}-\sum_{j=1}^{3} k_{j} R_{j}(m, \omega) e^{-k_{j} z}, \tag{52}
\end{align*}
$$

Substitution of Eqs. (31), (45), (51) and (52) into Eqs. (22)-(25), we get

$$
\begin{align*}
& \sigma_{x x}^{*}(z)=M_{1} R_{4}(m, \omega) e^{-k_{4} z}+\sum_{j=1}^{3} M_{3 j} R_{j}(m, \omega) e^{-k_{j} z}  \tag{53}\\
& \sigma_{z z}^{*}(z)=-M_{1} R_{4}(m, \omega) e^{-k_{4} z}+\sum_{j=1}^{3} M_{4 j} R_{j}(m, \omega) e^{-k_{j} z}, \tag{54}
\end{align*}
$$

$$
\begin{gather*}
\sigma_{x z}^{*}(z)=-M_{2} R_{4}(m, \omega) e^{-k_{4} z}-\sum_{j=1}^{3} M_{5 j} R_{j}(m, \omega) e^{-k_{j} z},  \tag{55}\\
p^{*}(z)=\sum_{j=1}^{3} M_{6 j} R_{j}(m, \omega) e^{-k_{j} z} . \tag{56}
\end{gather*}
$$

Where

$$
\begin{gathered}
\mathrm{M}_{1}=\frac{2 \mathrm{imk}_{4}\left(1-\beta_{1}\right)}{\alpha_{1}}, \quad \mathrm{M}_{2}=\frac{\left(\mathrm{k}_{4}^{2}+\mathrm{m}^{2}\right)\left(1-\beta_{1}\right)}{\alpha_{1}}, \quad \mathrm{M}_{3 \mathrm{j}}=\frac{1}{\alpha_{1}}\left[\left(2 \beta_{1}-1\right) \mathrm{k}_{\mathrm{j}}^{2}-\mathrm{m}^{2}-\mathrm{M}_{1 \mathrm{j}}-\mathrm{M}_{2 \mathrm{j}}\right] \\
\mathrm{M}_{4 \mathrm{j}}=\frac{1}{\alpha_{1}}\left[\mathrm{k}_{\mathrm{j}}^{2}-\left(2 \beta_{1}-1\right) \mathrm{m}^{2}-\mathrm{M}_{1 \mathrm{j}}-\mathrm{M}_{2 \mathrm{j}}\right], \quad \mathrm{M}_{5 \mathrm{j}}=\frac{2 \mathrm{im}\left(1-\beta_{1}\right) \mathrm{k}_{\mathrm{j}}}{\alpha_{1}}, \\
\mathrm{M}_{6 \mathrm{j}}=\frac{1}{\alpha_{1}}\left[-\mathrm{k}_{\mathrm{j}}^{2}+\mathrm{m}^{2}-\alpha_{1} \alpha_{2} \mathrm{M}_{1 \mathrm{j}}+\alpha_{1} \alpha_{4} \mathrm{M}_{2 \mathrm{j}}\right]
\end{gathered}
$$

The normal mode analysis is, in fact, to look for the solution in the Fourier transformed domain. Assuming that all the fields quantities are sufficiently smooth on the real line such that normal mode analysis of these functions exists.

## IV. THE BOUNDARY CONDITIONS

The non-dimensional boundary conditions on the surface $\mathrm{z}=0$ are:
(1) the concentrated load is suddenly applied normal to the free surface:

$$
\begin{equation*}
\sigma_{z Z}=-\mathrm{F}_{0} \exp (\omega \mathrm{t}+\mathrm{imx}) \tag{57}
\end{equation*}
$$

Where $\mathrm{F}_{0}$ is the normal load of intensity per unit length,
(2) the tangential stress component must be vanishing:

$$
\begin{equation*}
\sigma_{\mathrm{XZ}}=0 \tag{58}
\end{equation*}
$$

(3) there is no variation of concentration and temperature on the surface $z=0$,

$$
\begin{align*}
& \frac{\partial \mathrm{C}}{\partial \mathrm{z}}=0,  \tag{59}\\
& \frac{\partial \theta}{\partial \mathrm{z}}=0, \tag{60}
\end{align*}
$$

Substituting the expressions of the variables considered into the above boundary conditions, we can obtain the following equations satisfied by the parameters $\mathrm{R}_{\mathrm{j}}(\mathrm{j}=1,2,3,4)$

$$
\begin{aligned}
& \sum_{j=1}^{3} M_{4 j} R_{j}-M_{1} R_{4}=-F_{0}, \\
& \sum_{j=1}^{3} M_{5 j} R_{j}-M_{2} R_{4}=0, \\
& \sum_{j=1}^{3} M_{2 j} k_{j} R_{j}=0, \\
& \sum_{j=1}^{3} M_{1 j} k_{j} R_{j}=0,
\end{aligned}
$$

We obtain a system of four equations (61). After applying the inverse of matrix method,

$$
\left(\begin{array}{l}
R_{1}  \tag{62}\\
R_{2} \\
R_{3} \\
R_{4}
\end{array}\right)=\left(\begin{array}{lllc}
M_{41} & M_{42} & M_{43} & -M_{1} \\
M_{51} & M_{52} & M_{53} & -M_{2} \\
k_{1} M_{21} & k_{2} M_{22} & k_{3} M_{23} & 0 \\
k_{1} M_{11} & k_{2} M_{12} & k_{3} M_{13} & 0
\end{array}\right)^{-1}\left(\begin{array}{c}
-F_{0} \\
0 \\
0 \\
0
\end{array}\right)
$$

Solving the system of Eqs. (60), we get the parameters $\mathrm{R}_{\mathrm{j}}(\mathrm{j}=1,2,3,4)$, defined as follows:

$$
\mathrm{R}_{\mathrm{j}}=\frac{\Delta_{\mathrm{j}}}{\Delta}, \quad(\mathrm{j}=1,2,3,4)
$$

Where

$$
\begin{gathered}
\Delta=k_{1} k_{3} M_{2} M_{42}\left(M_{13} M_{21}-M_{11} M_{23}\right)+k_{1} k_{3} M_{1} M_{52}\left(M_{11} M_{23}-M_{13} M_{21}\right) \\
+k_{2} k_{3} M_{2} M_{41}\left(M_{12} M_{23}-M_{13} M_{22}\right)+k_{2} k_{3} M_{1} M_{51}\left(M_{13} M_{22}-M_{12} M_{23}\right), \\
+k_{2} k_{1} M_{2} M_{43}\left(M_{11} M_{22}-M_{12} M_{21}\right)+k_{2} k_{1} M_{1} M_{53}\left(M_{12} M_{21}-M_{11} M_{22}\right) \\
\Delta_{1}=F_{0} k_{2} k_{3} M_{2}\left(M_{13} M_{22}-M_{12} M_{23}\right), \\
\Delta_{2}=F_{0} k_{1} k_{3} M_{2}\left(M_{11} M_{23}-M_{13} M_{21}\right), \\
\Delta_{3}=F_{0} k_{2} k_{1} M_{2}\left(M_{12} M_{21}-M_{11} M_{22}\right), \Delta_{4}=F_{0} k_{2} k_{1} M_{53}\left(M_{12} M_{21}-M_{11} M_{22}\right) \\
+F_{0} k_{2} k_{3} M_{51}\left(M_{13} M_{22}-M_{12} M_{23}\right)+F_{0} k_{3} k_{1} M_{52}\left(M_{11} M_{23}-M_{13} M_{21}\right) . \\
\text { V. PARTICULAR CASE }
\end{gathered}
$$

By putting $C=0, a=0, b=0, \beta=0$, we get the equations for the displacements component, the stresses and the temperature without the effect of diffusion.

In this case, we obtain:

$$
\begin{gather*}
{\left[\mathrm{D}^{2}-\mathrm{m}^{2}-\alpha_{1} \omega^{2}\right] \varphi^{*}(\mathrm{z})-\theta^{*}(\mathrm{z})=0}  \tag{63}\\
{\left[\left(1-\beta_{1}\right)\left(\mathrm{D}^{2}-\mathrm{m}^{2}\right)-\alpha_{1} \omega^{2}\right] \psi^{*}(\mathrm{z})=0}  \tag{64}\\
\frac{\delta_{0} \delta \omega_{1}}{\alpha_{1}}\left(\mathrm{D}^{2}-\mathrm{m}^{2}\right) \varphi^{*}(\mathrm{z})-\left[\mathrm{D}^{2}-\mathrm{m}^{2}-\omega_{1}\right] \theta^{*}(\mathrm{z})=0 \tag{65}
\end{gather*}
$$

Eliminating $\varphi^{*}(\mathrm{z})$ and $\theta^{*}(\mathrm{z})$ between Eqs. (63) and (65), after some simple computations we get the following fourthorder ordinary differential equation satisfied with $\varphi^{*}(\mathrm{z})$ and $\theta^{*}(\mathrm{z})$ given by

$$
\begin{equation*}
\left[D^{4}-g_{11} D^{2}+g_{12}\right]\left\{\varphi^{*}(z), \theta^{*}(z)\right\}=0 \tag{66}
\end{equation*}
$$

Where

$$
\mathrm{g}_{10}=\frac{\delta_{0} \delta \omega_{1}}{\alpha_{1}}, \mathrm{~g}_{11}=2 \mathrm{~m}^{2}+\alpha_{1} \omega^{2}+\omega_{1}+\mathrm{g}_{10}, \mathrm{~g}_{12}=\mathrm{m}^{4}+\alpha_{1} \omega^{2} \mathrm{~m}^{2}+\mathrm{m}^{2} \omega_{1}+\alpha_{1} \omega_{1} \omega^{2}+\mathrm{m}^{2} \mathrm{~g}_{10}
$$

The solution of Eq. (66) is given by

$$
\begin{equation*}
\varphi^{*}(z)=\sum_{j=1}^{2} s_{j}(m, \omega) e^{-\lambda_{j} z} \tag{67}
\end{equation*}
$$

Where $\lambda_{j}(j=1,2)$ are the eigenvalues (roots) of the following characteristics equations

$$
\begin{equation*}
\lambda^{4}-\mathrm{g}_{11} \lambda^{2}+\mathrm{g}_{12}=0 \tag{68}
\end{equation*}
$$

Similarly

$$
\begin{equation*}
\theta^{*}(z)=\sum_{j=1}^{2} S_{j}^{\prime}(m, \omega) e^{-\lambda_{j} z} \tag{69}
\end{equation*}
$$

Where $\mathrm{s}_{\mathrm{j}}(\mathrm{m}, \omega)$ and $\mathrm{s}_{\mathrm{j}}^{\prime}(\mathrm{m}, \omega)$, are parameters depending on m and $\omega$.
Substituting Eqs. (67) and (64) into Eqs. (63)-(65), we get

$$
\begin{equation*}
S_{j}^{\prime}(m, \omega)=N_{1 j} S_{j}(m, \omega), \quad j=1,2 \tag{70}
\end{equation*}
$$

We thus have

$$
\begin{equation*}
\theta^{*}(\mathrm{z})=\sum_{\mathrm{j}=1}^{2} \mathrm{~N}_{\mathrm{ij}} \mathrm{~s}_{\mathrm{j}}(\mathrm{~m}, \omega) \mathrm{e}^{-\lambda_{\mathrm{j}} \mathrm{z}}, \tag{71}
\end{equation*}
$$

Where

$$
\mathrm{N}_{\mathrm{lj}}=\lambda_{\mathrm{j}}^{2}-\mathrm{m}^{2}-\alpha_{1} \omega^{2}
$$

The solution of Eq. (64) is the same in Eq. (33) and

$$
\begin{gather*}
u^{*}(z)=k_{4} R_{4}(m, \omega) e^{-k_{4} z}+i m \sum_{j=1}^{2} S_{j}(m, \omega) e^{-\lambda_{j} z},  \tag{72}\\
w^{*}(z)=i m R_{4}(m, \omega) e^{-k_{4} z}-\sum_{j=1}^{2} \lambda_{j} S_{j}(m, \omega) e^{-\lambda_{j} z},  \tag{73}\\
\sigma_{x x}^{*}(z)=M_{1} R_{4}(m, \omega) e^{-k_{4} z}+\sum_{j=1}^{2} N_{2 j} S_{j}(m, \omega) e^{-\lambda_{j} z},  \tag{74}\\
\sigma_{z z}^{*}(z)=-M_{1} R_{4}(m, \omega) e^{-k_{4} z}+\sum_{j=1}^{2} N_{3 j} S_{j}(m, \omega) e^{-\lambda_{j} z},  \tag{75}\\
\sigma_{x z}^{*}(z)=-M_{2} R_{4}(m, \omega) e^{-k_{4} z}-\sum_{j=1}^{2} N_{4 j} S_{j}(m, \omega) e^{-\lambda_{j} z}, \tag{76}
\end{gather*}
$$

Where

$$
\begin{gathered}
\mathrm{N}_{2 \mathrm{j}}=\frac{1}{\alpha_{1}}\left[\left(2 \beta_{1}-1\right) \lambda_{\mathrm{j}}^{2}-\mathrm{m}^{2}-\mathrm{N}_{1 \mathrm{j}}\right], \\
\mathrm{N}_{3 \mathrm{j}}=\frac{1}{\alpha_{1}}\left[\lambda_{\mathrm{j}}^{2}-\left(2 \beta_{1}-1\right) \mathrm{m}^{2}-\mathrm{N}_{1 \mathrm{j}}\right], \\
\mathrm{N}_{4 \mathrm{j}}=\frac{2 \mathrm{im}\left(1-\beta_{1}\right) \lambda_{\mathrm{j}}}{\alpha_{1}}, \quad \mathrm{j}=1,2 .
\end{gathered}
$$

In this case, the non-dimensional boundary conditions on the surface $\mathrm{z}=0$ are:

$$
\begin{equation*}
\sigma_{z z}=-\mathrm{F}_{0} \exp (\omega \mathrm{t}+\mathrm{imx}), \sigma_{\mathrm{Xz}}=0, \frac{\partial \theta}{\partial \mathrm{z}}=0, \text { on } \mathrm{z}=0 \tag{77}
\end{equation*}
$$

Substituting the expressions of the variables considered into the above boundary conditions, we can obtain the following equations satisfied by the parameters $\mathrm{S}_{\mathrm{j}}(\mathrm{j}=1,2)$ and $\mathrm{R}_{4}$

$$
\begin{align*}
& \sum_{j=1}^{2} N_{3 j} S_{j}-M_{1} R_{4}=-F_{0}, \\
& \sum_{j=1}^{2} N_{4 j} S_{j}-M_{2} R_{4}=0,  \tag{78}\\
& \sum_{j=1}^{2} N_{1 j} \lambda_{j} S_{j}=0 .
\end{align*}
$$

Solving the above system of Eqs. (78), we get the parameters $S_{j}(j=1,2)$ and $R_{4}$ in the following forms respectively:

$$
\mathrm{S}_{\mathrm{j}}=\frac{\Delta_{\mathrm{j}}^{*}}{\Delta^{*}}, \quad(\mathrm{j}=1,2), \quad \mathrm{R}_{4}=\frac{\Delta_{3}^{*}}{\Delta^{*}}
$$

Where

$$
\begin{gathered}
\Delta^{*}=\lambda_{1} \mathrm{~N}_{11}\left(\mathrm{M}_{1} \mathrm{~N}_{42}-\mathrm{M}_{2} \mathrm{~N}_{12}\right)+\lambda_{2} \mathrm{~N}_{11}\left(\mathrm{M}_{2} \mathrm{~N}_{31}-\mathrm{M}_{1} \mathrm{~N}_{41}\right), \\
\Delta_{1}^{*}=-\mathrm{F}_{0} \lambda_{2} \mathrm{~N}_{12} \mathrm{M}_{2}, \Delta_{2}^{*}=\mathrm{F}_{0} \lambda_{1} \mathrm{~N}_{11} \mathrm{M}_{2}, \\
\Delta_{3}^{*}=\mathrm{F}_{0}\left(\lambda_{1} \mathrm{~N}_{11} \mathrm{~N}_{42}-\lambda_{2} \mathrm{~N}_{12} \mathrm{~N}_{41}\right) .
\end{gathered}
$$

## VI. NUMERICAL RESULTS AND DISCUSSIONS

The copper material was chosen for the purpose of numerical example. Since, we have $\omega=\omega_{0}+i \xi$, where $i$ is the imaginary unit, $\mathrm{e}^{\omega \mathrm{t}}=\mathrm{e}^{\omega_{0} \mathrm{t}}(\cos \xi \mathrm{t}+\mathrm{i} \sin \xi \mathrm{t})$, and for small values of time, we can take $\omega=\omega_{0}$ (real).

The numerical constants of the problems were taken as (in SI unit):

$$
\begin{gathered}
\lambda_{0}=0.5 \times 10^{11}, \quad \mu_{0}=3.86 \times 10^{11}, \quad \beta_{0}=0.1, \quad v_{0}=0.3 \times 10^{-2}, \quad \alpha_{t}=1.78 \times 10^{5}, \rho=8954 \\
\alpha_{c}=1.98 \times 10^{-4}, \mathrm{a}=1.2 \times 10^{4}, \mathrm{~b}=0.9 \times 10^{6}, \mathrm{~d}=0.85 \times 10^{-8}, \mathrm{~T}_{0}=293, \tau_{0}=0.02, \omega_{0}=2.5 \\
\mathrm{C}_{\mathrm{E}}=383.1, \alpha^{*}=0.002, \mathrm{~m}=1.8, \mathrm{~F}_{0}=0.1
\end{gathered}
$$

In the present work, numerical calculations are carried out in two different cases. In the first case, we are investigating how the non-dimensional displacement components, the temperature, the stress components and the chemical potential vary with different values of the fractional parameter $\alpha$ against z in the presence and absence of diffusion when the time remain constant. In the second case, we will show how the non-dimensional displacement components, the temperature, the stress components and the chemical potential vary with different values of $\alpha_{1}$ against $z$ when the time instant remain constant. The computations are carried out at $x=2.2$ for the time instant $t=0.1$ in the range $0 \leq z \leq 10$. The numerical results of the real parts of all the physical quantities are obtained and presented graphically in figs. 1-14 of the above three different cases. The figs. 15-21 as the first case but in 3D graphic.

Figs. 1-7 depict the variety of the displacement components $\mathrm{u}, \mathrm{w}$ the temperature $\theta$, the stress components $\sigma_{\mathrm{xx}}, \sigma_{\mathrm{xz}}$ the chemical potential P and concentration of the diffusive material C for two different values of the fractional parameter $\alpha$, namely for $\alpha=1.0$ and $\alpha=0.5$ in the presence and absence of the diffusion effect. Figs. 1 and 2 show that for all the cases, $u$ and $w$ remain close to the zero value in the considered domain of the distance $z$ far from the origin, except near the vicinity of the load where slight variations are noticed. It is also clearly depicted from figs. 1 and 2 that the values of the displacement components u and w are maximum in the thermoelastic medium without diffusion effect for $\alpha=1.0$. Fig. 3 clearly show that the range of magnitude of the temperature $\theta$ is greater in the thermoelastic medium with diffusion effect than that in thermoelastic medium without this effect. Fig. 4, 6,7 shows that the value of the stress $\sigma_{\mathrm{xx}}$ is maximum in the thermoelastic medium without diffusion effect for $\alpha=1.0$. The chemical potential P and concentration of the diffusive material C is maximum in the thermoelastic medium with diffusion effect for $\alpha=1.0$ while from fig. 5 , we see that the value of the stress $\sigma_{\mathrm{xz}}$ is maximum in the thermoelastic medium with diffusion effect for $\alpha=1.0$.


Fig. 1 Variation of displacement distribution $u$ at $\alpha_{1}=1.6$


Fig. 2 Variation of displacement distribution w at $\alpha_{1}=1.6$


Fig. 3 Variation of temperature distribution w at $\alpha_{1}=1.6$


Fig. 4 Stress distribution $\sigma_{\mathrm{xx}}$ at $\alpha_{1}=1.6$


Fig. 5 Stress distribution $\sigma_{\mathrm{xz}}$ at $\alpha_{1}=1.6$


Fig. 6 Variation of chemical potential P at $\alpha_{1}=1.6$


Fig. 7 Variation of concentration of the diffusive material C at $\alpha_{1}=1.6$
Figs. 8-13 display the distribution of the displacement functions $u$, $w$ the temperature $\theta$, the stress functions $\sigma_{\mathrm{xx}}, \sigma_{\mathrm{xz}}$, the chemical potential P and concentration of the diffusive material C for two different values of the parameter $\alpha_{1}$, namely for $\alpha_{1}=1.0$ (temperature independent modulus of elasticity) and $\alpha_{1}=1.6$ (temperature dependent modulus of elasticity) in the presence and absence of the diffusion effect.

Figs. 8 and 9 exhibit that the values of the displacement components $u$ and $w$ are maximum in the thermoelastic medium without diffusion effect when the modulus of elasticity is temperature dependent. The values of the displacement functions $u$ and $w$ vanish after $z>10$ (approximately). Figs. 10 exhibits that the range of magnitudes of $\theta$ is greater in the thermoelastic medium with diffusion effect than that in thermoelastic medium without this effect. It is also clearly depicted from figs. 10 that the values of $\theta$ are maximum in the thermoelastic medium with diffusion effect when the modulus of elasticity is temperature dependent. Also these values approach to the zero value more rapidly with distance z in the thermoelastic medium with diffusion effect than that in thermoelastic medium without this effect. Figs. 11 and 12 display that the values of the stresses $\sigma_{\mathrm{xx}}$ and $\sigma_{\mathrm{xz}}$ are maximum in the thermoelastic medium with diffusion effect for when the modulus of elasticity of the medium is temperature independent. Figs. 13 and 14 shows that the value of the chemical potential P and concentration of the diffusive material C is minimum in the thermoelastic medium with diffusion effect for $\alpha_{1}=1.0$. Also it can be noticed that the values of the stress functions approach to zero more rapidly in the case of the presence of a diffusion effect than in the case of the absence of a diffusion effect with the distance z increases.


Fig. 8 Variation of displacement distribution u at $\alpha=0.5$


Fig. 9 Variation of displacement distribution w at $\alpha=0.5$


Fig. 10 Variation of temperature distribution $\theta$ at $\alpha=0.5$


Fig. 11 Stress distribution $\sigma_{\mathrm{xx}}$ at $\alpha=0.5$


Fig. 12 Stress distribution $\sigma_{\mathrm{xz}}$ at $\alpha=0.5$


Fig. 13 Variation of chemical potential P at $\alpha=0.5$


Fig. 14 Variation of concentration of the diffusive material C at $\alpha=0.5$

Figs. 15-21 are giving 3D surface curves for the physical quantities i.e., the displacement components, the temperature, the stress components and the chemical potential vary with the value of the fractional parameter $\alpha=0.5$ and $\alpha_{1}=1.6$ in the presence of diffusion when the time remains constant.


Fig. 15 Displacement distribution $u$ against both components of distance


Fig. 16 Displacement distribution w against both components of distance


Fig. 17 Temperature distribution $\theta$ against both components of distance


Fig. 18 Stress distribution $\sigma_{\mathrm{xx}}$ against both components of distance


Fig. 19 Stress distribution $\sigma_{\mathrm{xz}}$ against both components of distance


Fig. 20 Chemical potential P against both components of distance


Fig. 21 Variation of concentration of the diffusive material C against both components of distance
These figures are very important to study the dependence of these physical quantities on vertical component of distance. The curves obtained are highly depending on vertical distance from origin, some quantities increases on negative direction of vertical distance while some on positive direction of vertical displacement.

## VII. CONCLUSION

According to the analysis above and from the numerical results presented in figs. 1-21, we can conclude the following important points:
(i) The presence of diffusion plays a significant role in all the quantities and has an important effect on the vertical and normal components of displacement, the temperature, the stress components and the chemical potential.
(ii) It was observed that the dependence of the modulus of elasticity on the reference temperature $\left(\alpha_{1}\right)$ plays a significant role in the thermal interactions, while the presence of the modulus of elasticity on reference temperature has a significant effect in all the physical quantities. The important point of this work is the consideration that the temperature depends on the material properties, while in other works these material properties were assumed to be constant. This study is very important for microscale problems, because in these cases the material parameters are temperature dependent.
(iii) The method used in the present article is applicable to a wide range of problems in thermodynamics Othman e $t$ al (2005; 2008a 2008b; 2012a; 2011; 2012b; 2012c).
(iv) The problem considering the effect of diffusion in generalized thermoelasticity with dependence of the modulus of elasticity on the reference temperature $\left(\alpha_{1}\right)$ can be described by characteristic equations of sixth order.

## REFERENCES

[1] Caputo,M., Mainardi, F., (1971a) A new dissipation model based on memory mechanism. Pure Appl. Geophys. 91, 134 -147.
[2] Caputo,M., Mainardi, F., (1971b) Linear models of dissipation in an elastic solid. Rivis ta Del Nuovo Cimento. 1, 161-198.
[3] Caputo,M., (1974) Vibrations of an infinite viscoelastic layer with a dissipative memory. J. Acoust. Soc. Am. 56, 897-904.
[4] Caputo,M., (1967) Linear model of dissipation whose Q is almost frequency independent-II. Geophys. J. Roy. Astron. Soc. 13, 529-39.
[5] Ezzat, M. A., Fayik, M. A., (2011) Fractional order theory of thermoelastic diffusion. J. Thermal Stresses. 34, 851-872.
[6] Fryxel, R.E., Aitken, E.A., (1969) High temperature studies of uranium in a thermal gradient. J. Nucl. Mater. 30, 50-56.
[7] Ignaczak, J., Ostoja-Starzewski, M., (2010) Thermoelasticity with finite wave speeds. Oxford University Press, New York.
[8] Kimmich, R., (2002) Strange Kinetics, Porous media and NMR. J. Chem. Phys. 284, 253-285.
[9] Mainardi, F., (1998) Applications of fractional calculus in mechanics. in: P. Rusev, I. Dimovski, V. Kiryakova (Eds.), Transforms Method and Special Functions. Bulgarian Academy of Sciences, Sofia, 309-334.
[10] Nowacki,W., (1974a) Dynamic problems of thermoelastic diffusion in solids. J. I Bulletin de l’ Acadermic Polonaise des Sciences Serie des Sciences Techniques. 22, 55-64.
[11] Nowacki,W., (1974b) Dynamic problems of thermoelastic diffusion in solids. J. II Bulletin de l' Acadermic Polonaise des Sciences Serie des Sciences Techniques. 22, 129-135.
[12] Nowacki,W., (1974c) Dynamic problems of thermoelastic diffusion in solids. J. III Bulletin de l' Acadermic Polonaise des Sciences

Serie des Sciences Techniques. 22, 266-275.
[13] Nowacki,W., (1976) Dynamic problems of diffusion in solids. J. Eng. Frac. Mech. 8, 261266.
[14] Oriani, R.A., (1969) Thermomigration in solid metals. J. Phys.Chem. 30, 339-351.
[15] Othman, M. I. A., Sarhan,Y. Atwa, Farouk, R., (2009) The effect of diffusion on two-dimensional problem of generalized thermoelasticity with Green-Naghdi theory. Int. Commu. Heat Mass Trans. 36, 857-864.
[16] Othman, M. I. A., (2005) Effect of rotation and relaxation time on a thermal shock problem for a half-space in generalized thermoviscoelasticity. Acta Mech. 174, 129-143.
[17] Othman, M. I. A., Song,Y.Q., (2008) Effect of rotation on plane waves of the generalized electromagneto-thermo-viscoelasticity with two relaxation times. Appl. Math. Modell. 32, 811-825.
[18] Othman, M. I. A., Sarhan,Y. Atwa, . Farouk, R., (2008) MGeneralized magneto-thermo-viscoelastic plane waves under the effect of rotation without energy dissipation. Int. J. Eng. Science. 46, 639-653.
[19] Othman, M. I. A., Sarhan, Y. Atwa, Farouk, R. M., (2012a) Response of micropolar thermoelastic medium with voids due to various sources in thermo-elasticity III. Acta Mechanica Solida Sinica. 25, 197-209.
[20] Othman, M. I. A., Sarhan, Y. Atwa., (2011) The Effect of magnetic field on 2-D problem of generalized thermoelasticity without energy dissipation. Int. J. Industrial Mathematics. 3, 213-226.
[21] Othman, M. I. A., Sarhan,Y. Atwa., (2012b) Thermoelastic plane waves for an elastic solid half-space under hydrostatic initial stress of type III. 47, 1337-1347.
[22] Othman, M. I. A., Sarhan, Y. Atwa., (2012c) Generalized magneto-thermo-elasticity in a fibre-reinforced anisotropic half-space without energy dissipation. Int. J. of thermophysics. 33, 1126-1142.
[23] Povstenko,Y. Z., (2005) Fractional heat conduction equation and associated thermal stress. J. Therm. Stresses. 28, 83-102.
[24] Rabotnov, Yu. N., (1966) Creep of structural elements [in Russian]. Nauka, Moscow.
[25] Sherief, H., Hamza, F., Saleh, H., (2004) The theory of generalized thermoelastic diffusion. Int. J. Eng. Sci. 42, 591-608.
[26] Sherief, H., Saleh, H., (2005) A half-space problem in the theory of generalized thermoelastic diffusion. Int. J. Solids Struct. 42, 44844493.
[27] Singh, B., (2005) Reflection of P and SV waves from free surface of an elastic solid with generalized thermodiffusion. J. Earth. Syst. Sci. 114, 159-168.
[28] Thomas, L., (1980) Fundamentals of Heat Transfer [M], Prentice-Hall Inc, Englewood Cliffs, New Jersey.

