Exponentially Expanding Universe from Minisuperspace Wheeler-De Witt Equation

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***Abstract*- The minisuperspace Wheeler-De Witt equation is solved exactly for some suitable parameters. A slightly modified Wheeler-De Witt equation has been considered to examine the effect of higher order field variables on the potential function, through which an exact solution with suitable parameters is found. From the solution, it is seen that the universe expands or contracts exponentially. Other physical significance of the solution is also discussed.**

Keywords*- *Quantum cosmology; Wheeler-De Witt equation; Minisuperspace; Exact solution

I. INTRODUCTION

Three fundamental forces of nature, namely, electromagnetic, weak nuclear and strong nuclear forces, have been successfully explained with the help of quantum theory. But it has not been possible yet to bring the gravitational force into this scheme. The quantization of gravitation is a very difficult problem, although there have been simplified approaches to handle the problem. One of these approaches is quantum cosmology which comes out from the marriage between general relativity and quantum theory. An aspect of this approach is the consideration of a closed model of the universe with finite time duration and compact space-like sections with three geometries h_{ij}. One can write down a functional differential equation for the wave function of the universe $\Psi[h_{ij}]$, which is a functional of h_{ij}. This is the so-called Wheeler-De Witt equation (WDW) [1] given by:

$$\mathcal{H}\Psi[h_{ij}] = \left\{-G_{ijkl}\,\frac{\partial^2}{\partial h_{ij}\,\partial h_{kl}} - \sqrt{h}\;{}^3R\right\}\Psi[h_{ij}] = 0 \tag{1}$$

where G_{ijkl} is known as the DeWitt metric or (5+1) dimensional metric on superspace with signature (-+++++), given by

$$G_{ijkl} = \frac{1}{2\sqrt{h}}\left(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}\right) \tag{2}$$

${}^3R$ is the scalar curvature of the intrinsic geometry of three-surface and h is the determinant of the metric h_{ij}. The inverse of DeWitt metric can be given by:

$$G^{ijkl} = \frac{\sqrt{h}}{2}\left(h^{ik}h^{jl} + h^{il}h^{jk} - h^{ij}h^{kl}\right) \tag{3}$$

Now it is clear from Eqs. (2) and (3) that

$$G^{ijkl}\,G_{klmn} = \frac{1}{2}\left(\delta^i_m\delta^j_n + \delta^i_n\delta^j_m\right) \tag{4}$$

The WDW equation is a ‘hyperbolic’ functional differential equation in superspace [2]. In the three-space with metric h_{ij}, for example, the h_{ij} is the function of x^1, x^2, x^3 and determines the *distance* $d\sigma$ between infinitesimally separated points$(x^1,\ x^2,\ x^3)$ and $(x^1+\Delta x^1,\ x^2+\Delta x^2,\ x^3+\Delta x^3)$ as follows:

$$d\sigma^2 = h_{ij}\,\Delta x^i\Delta x^j \tag{5}$$

This distance is invariant under spatial coordinate transformations. In a similar manner, the space of all matrices h_{ij} can be regarded as a *superspace* in which the *points* are the metric functions of h^{ij}, (which are the inverse of h_{ij}), and one can define a corresponding metric G_{ijkl} so that the distance $d\Sigma$ between neighboring metric h^{ij} and $h^{ij}+\delta h^{ij}$ is given by

$$d\Sigma^2 = G_{ijkl}\,h^{ij}\,h^{kl} \tag{6}$$

which is invariant in a suitable sense when one transforms one set of h^{ij} to another set h'^{ij}. The DeWitt metric can be regarded as 6×6 matrix in the symmetric space of (ij) . So Eq. (6) can be written as

$$d\Sigma^2 = G_{AB}\,\delta h^A\delta h^B, \qquad A,B = 1,2,\dots,6, \tag{7}$$

$$A,B \in \{h_{11},h_{12},h_{13},h_{22},h_{23},h_{33}\} \equiv \{h_1,h_2,h_3,h_4,h_5,h_6\}. \tag{8}$$

II. MINISUPERSPACE WHEELER-DE WITT EQUATION

The solution of WDW equation [3-10] in its full form is very hard for its infinite dimensionality. To make the problem tractable, one can simplify the WDW equation by restricting it to *minisuperspace*, that is, by considering some limited fluctuations of the geometry to contribute to the sum of histories in the path integral, such as geometries conformal to any given geometry. In this way we get some standard and suitably modified differential equations or simplified functional differential equations which might be solved exactly and might be given some suitable interpretations.

Let us consider a "minisuperspace" which is defined by homogeneous and isotropic manifolds where Euclidean histories can contribute to the sum defining the wave function. The suitable metric of this form can be given by

$$ds^2 = \sigma\left[N^2(t)dt^2 + a^2(t)d\Omega_3^2\right] \tag{9}$$

where $N(t)$ is the lapse function, $\sigma^2 = l^2/24\pi^2$, $l^2 = 16\pi = 2\kappa^2$ and $d\Omega_3$ is the line element on the three- sphere [2,11], S_3, given by

$$d\Omega_3^2 = d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta \ d\phi^2) \tag{10}$$

The Euclidean action can be written by the following equation

$$l^2 I_E = -\int_V d^4x \, g^{\frac{1}{2}}({}^4R - 2\Lambda) + 2\int_{\partial V} d^3x \, h^{\frac{1}{2}}K^2 \tag{11}$$

The corresponding minisuperspace action is given by

$$I_E = \frac{1}{2}\int d\tau \left(\frac{N}{a}\right)\left[-\left(\frac{a\dot{a}}{N}\right)^2 - a^2 + \lambda a^4\right] \tag{12}$$

where $\frac{\sigma^2\Lambda}{3} = H^2 = \lambda$, $\dot{a}$ = da/dτ, H is the Hubble constant and λ is a parameter. Now the WDW can be given by an ordinary partial differential equation of the form:

$$\frac{1}{2}\left[\frac{1}{a^p}\frac{\partial}{\partial a}\left[a^p\frac{\partial}{\partial a}\right] - a^2 + \lambda a^4\right]\Psi(a) = 0 \tag{13}$$

where p is a suitable parameter. The wave function of the universe $\Psi(a)$ is given by a path integral over all compact metrices of the form (9) which are bound by a three-sphere of radius a.

Let us now introduce a conformally invariant scalar field φ to represent the matter degrees of freedom. Thus one obtains the action keeping χ and a fixed on the boundary

$$\mathrm{I} = \frac{1}{2}\int \mathrm{dt}\left[\frac{\mathrm{N}}{\mathrm{a}}\right]\left[\left[\frac{\mathrm{a}}{\mathrm{N}}\dot{\mathrm{a}}\right]^2 + \mathrm{a}^2 - \lambda\mathrm{a}^4 + \left[\frac{\mathrm{a}}{\mathrm{N}}\dot{\chi}\right] - \chi^2\right] \tag{14}$$

where χ is conformally invariant and related to ϕ by the following relation:

$$\phi = \frac{\chi}{(2\pi^2\sigma^2)^{\frac{1}{2}}\,a} \tag{15}$$

Therefore the corresponding minisuperspace WDW equation can be given by

$$\frac{1}{2}\left[\frac{1}{a^p}\frac{\partial}{\partial a}\left[a^p\frac{\partial}{\partial a}\right] - a^2 + \lambda a^4 - \frac{\partial^2}{\partial\chi^2} + \chi^2 - 2\epsilon_0\right]\Psi(a,\chi) = 0 \tag{16}$$

where $2\epsilon_0$ is an arbitrary constant included in the matter-energy renormalization [5]. Thus, an infinite number of degrees of freedom of the model can be reduced only to two $a(t)$ and $\phi(t)$, where $a(t)$ is the scale factor. The scale factor can be regarded as the radius of the known universe.

III. AN EXACT SOLUTION OF MINISUPERSPACE WDW EQUATION

In this paper, we are only interested in exact solutions of WDW equation. We have found an exact solution with the parameters $p = 1, \lambda = 0$. For $\lambda = 0$, the cosmological constant $\Lambda = 0$, and consequently, the Hubble constant $H = 0$, too. This is expected for the ground state of the universe. Hence it seems that the expansion of the universe is not possible in this case. But if one carefully looks at the equations (15) and (16), it will be evident that the term ϕ^2 serves as an effective cosmological constant. Therefore, the scalar field will initiate expansion or contraction. Let us put $\chi = i\xi$, and $r^2 = a^2 + \chi^2$. Now the equation (16) becomes

$$\frac{1}{a}\frac{\partial\Psi}{\partial a} + \frac{\partial^2\Psi}{\partial a^2} + \frac{\partial^2\Psi}{\partial\xi^2} - (r^2 + 2\epsilon_0)\Psi = 0 \tag{17}$$

The first three terms in Eq. (17) can be considered as a two dimensional *Laplacian* in the variables a and ξ. Hence we can write Eq. (17) as follows:

$$\nabla^2\Psi - (r^2 + 2\epsilon_0)\Psi = 0 \tag{18}$$

Again, let $\Psi = F(r)$ and Eq. (18) leads to

$$F'' + \frac{2F'}{r} = (r^2 + 2\epsilon_0)F \tag{19}$$

Put $F = A\exp(\alpha r^2)\, u$, where A and α are arbitrary constants, by which, Eq. (19) becomes

$$2\alpha u + 4\alpha^2 r^2 u + 4\alpha r u' + u'' + 4\alpha u + \frac{2}{r}u' = (r^2 + 2\epsilon_0)u \tag{20}$$

Again put $6\alpha = 2\epsilon_0$, $4\alpha^2 = 1$, then

$$u'' + \left(4\alpha r + \frac{2}{r}\right)u' = 0 \tag{21}$$

The exact solution of Eq. (21) can be given by

$$u = B\int^r \frac{1}{r^2}\exp(-2\alpha r^2)\, dr \tag{22}$$

where B is some arbitrary constant. Hence the exact solution of Eq. (19) is given by

$$F(r) = AB\exp(\alpha r^2)\int^r \frac{\exp(-2\alpha r^2)}{r^2}\, dr \tag{23}$$

Therefore,

$$\Psi(r) = F(r) = C\exp(\alpha r^2)\int^r \frac{\exp(-2\alpha r^2)}{r^2}\, dr \tag{24}$$

where $AB = C$, which is also a constant. This then provides one exact solution of the “minisuperspace” Wheeler-De Witt equation. The wave function $\Psi(r)$ in Eq. (24) contains an integral which does not exist for infinite limiting values; but for suitable finite limiting values, it yields a constant. Considering the integral yields a constant, the wave function becomes an exponential function of r which is depicted in Figs. 1 and 2 for values $\alpha = +1/2$ and for $\alpha = -1/2$, respectively:

$$\Psi(r) = D\exp\left(\pm\frac{1}{2}r^2\right) \tag{25}$$

where D is a constant. The physical significance of this solution will be discussed in the discussion.

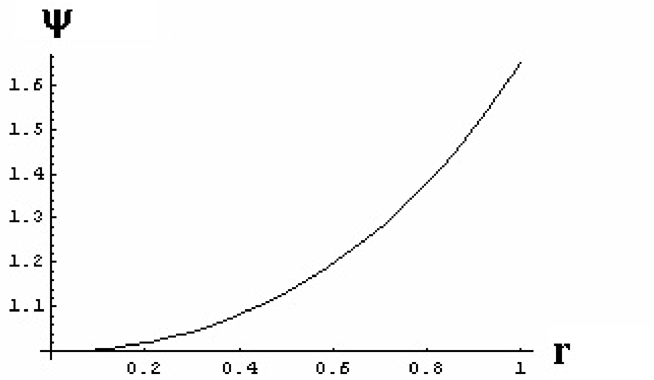

Fig. 1 Graphical representation of Eq. (25) with suitable parameters for $\alpha = +1/2$

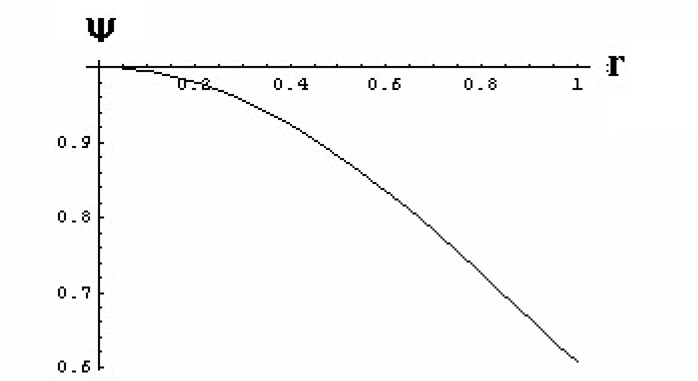

Fig. 2 Graphical representation of Eq. (25) with suitable parameters for $\alpha = -1/2$

IV. MODIFIED WDW EQUATION

After the Big Bang when the universe starts expansion, the fields are highly non-linear in nature for extreme spatial curvature. The potential function in Eq. (16) is in its simplest form. One can try for higher order terms of the fields to investigate the higher order effects. With this motivation to get an exact solution, we include the term $\eta(\chi^2 - a^2)^3$ in the potential function of WDW equation and rename the equation as modified WDW equation. Therefore, the modified WDW equation can be given by

$$\left[\frac{1}{a^p}\frac{\partial}{\partial a}\left[a^p \frac{\partial}{\partial a}\right] - a^2 + \lambda a^4 - \frac{\partial^2}{\partial \chi^2} + \chi^2 - 2\epsilon_0 + \eta(\chi^2 - a^2)^3\right]\Psi(a,\chi) = 0 \tag{26}$$

where η is an arbitrary parameter. The potential function is depicted in Figs. 3 and 4 for different suitable values of the parameter η.

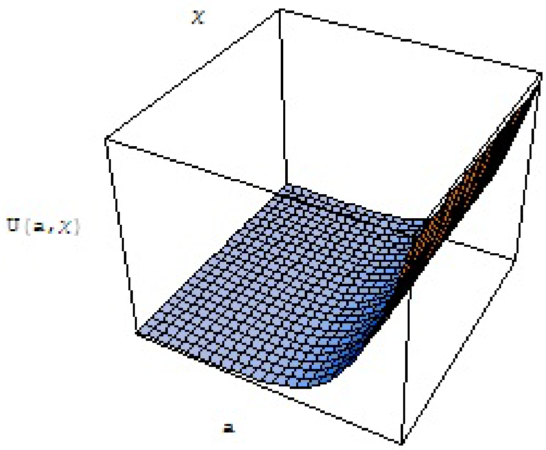

Fig. 3 3D graphical representation of modified WDW potential ($\lambda = 0$) with suitable parameters for $\eta = +ve$ values

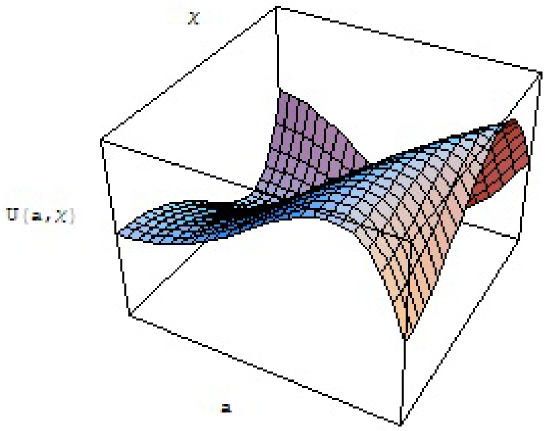

Fig. 4 3D graphical representation of modified WDW potential ($\lambda = 0$) with suitable parameters for $\eta = -ve$ values

V. AN EXACT SOLUTION OF MODIFIED WDW EQUATION

We put $(\chi^2 - a^2) = \zeta$. Therefore it can be written that $\Psi(a,\chi) = \Psi(\zeta)$. Now we have

$$\frac{\partial \Psi}{\partial a} = \frac{\partial \Psi}{\partial \zeta}\frac{\partial \zeta}{\partial a} = -2a\frac{\partial \Psi}{\partial \zeta} \equiv -2a\Psi' \tag{27}$$

$$\frac{\partial^2 \Psi}{\partial a^2} = 4a^2\frac{\partial^2 \Psi}{\partial \zeta^2} - 2\frac{\partial \Psi}{\partial \zeta} \equiv 4a^2\Psi'' - 2\Psi' \tag{28}$$

$$\frac{\partial \Psi}{\partial \chi} = \frac{\partial \Psi}{\partial \zeta}\frac{\partial \zeta}{\partial \chi} = 2\chi\frac{\partial \Psi}{\partial \zeta} \equiv 2\chi\Psi' \tag{29}$$

$$\frac{\partial^2 \Psi}{\partial \chi^2} = 4\chi^2\frac{\partial^2 \Psi}{\partial \zeta^2} + 2\frac{\partial \Psi}{\partial \zeta} \equiv 4\chi^2\Psi'' + 2\Psi' \tag{30}$$

Putting all these in Eq. (26) and considering $= 0$, we have

$$-4\zeta\Psi'' - (4+2p)\Psi' + (\zeta - 2\epsilon_0 + \sigma\zeta^3)\Psi = 0 \tag{31}$$

We consider a trial solution of (31)

$$\Psi = (J_0 + J_1\zeta)\exp(-\frac{1}{2}\kappa\zeta^2) \tag{32}$$

where J_0, J_1 and κ are arbitrary constants. Calculating Ψ' and Ψ'' and putting these in Eq. (31) and equating the powers of ζ, we get

$$\zeta^0:\ -(4+2p)J_1 - 2\epsilon_0 J_0 = 0 \tag{33}$$

$$\zeta:\ (4+2p)\kappa J_0 + 4\kappa J_0 - 2\epsilon_0 J_1 + J_0 = 0 \tag{34}$$

$$\zeta^2: (4+2p)\kappa J_1 + 12\kappa J_1 + J_1 = 0 \tag{35}$$

$$\zeta^3: -4\kappa^2 J_0 + J_0\eta = 0 \tag{36}$$

$$\zeta^4: -4\kappa^2 J_1 + J_1\eta = 0 \tag{37}$$

Solving Eqs. (33-37), we get

$$\frac{J_1}{J_0} = \frac{4}{\epsilon_0(16+2p)} = \frac{-2\epsilon_0}{(4+2p)} \tag{38}$$

Therefore,

$$\epsilon_0^2 = \frac{-2(4+2p)}{(16+2p)} \tag{39}$$

Let $p = -4$, then we accordingly get $\epsilon_0 = \pm 1$, $\kappa = -1/8$ and $\eta = 1/16$. Therefore the exact solution is given by

$$\Psi = A(2 + \chi^2 - a^2)\exp\left[\frac{1}{16}(\chi^2 - a^2)^2\right] \tag{40}$$

where A is a normalizing constant. Now, putting $(\chi^2 - a^2) = r^2$ in Eq. (40) and we get

$$\Psi(r) = A(2 + r^2)\exp(r^4/16) \tag{41}$$

VI. DISCUSSIONS

The exact solution of WDW equation given by Eq. (25) indicates that the universe starts expansion from a minimum finite radius and expands exponentially up to a maximum finite radius (for $\alpha = +1/2$) and bouncing back begins to contract exponentially up to a minimum radius (for $\alpha = -1/2$). Therefore the universe oscillates between finite radii. Hartle-Hawking model [5] represents a universe expanding from zero radius up to a maximum value and then collapsing back to zero radius. There is also a small probability that the universe might tunnel through the barrier to indefinite expansion in this present model and Hartle-Hawking model. The exact solution of Eq. (24) looks interesting from different points of view. Let us rewrite the Eq. (24) as follows:

$$\Psi(r)\exp(-\alpha r^2) = C\int^r \frac{\exp(-2\alpha r^2)}{r^2}dr \tag{42}$$

By differentiating and rearranging, we get

$$r^2\Psi'(r) - 2\alpha r^3\Psi(r) = C\,e^{-\alpha r^2} \tag{43}$$

From (43), it is readily seen that as $r \to \infty$, then $\Psi \to 0$. Hence it is consistent to assume that the universe is asymptotically flat. This might explain the fact that why the present universe is flat.

From Eq. (41), when $r = 0$ then $\Psi = constant$, i.e., the universe starts expansion from a zero radius and expands exponentially for $\chi \neq a$. But when $\chi = a$, then $\Psi = constant$ again, i.e., and the universe becomes steady. Hence it is observed that the universe expands from a zero radius up to a certain value and ultimately becomes steady and it never contracts, i.e., there is a Big Bang, but not a Big Crunch. There is also the possibility that the stationary universe might transit to the situation where it will be expanding forever.

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