Towards the Understanding of Turbulence

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Abstract- **In a previous paper, the author proposed turbulence as a second order phase transition with spontaneous broken symmetry. This has been experimentally confirmed by research groups from the University of Illinois and Weizemann Institute of Science. In this paper we propose turbulence as a classical analog of Bose Einstein condensate and the Gross-Pitaevskii equation is used to derive the condensation free energy. The critical value of the order parameter, the condensation wave function is determined. This is the value where turbulence occurs and spontaneous symmetry breaking in the ground state of the condensation free energy takes place. Being a condensate, there is molecular pairing in turbulence. We determine the numerical value of the condensation free energy with the use of the coupled oscillation model for the pair of molecules. Our expression for the condensation free energy also yields a power series in terms of the order parameter in agreement with the Landau phenomenology of second order phase transition. This confirms that turbulence is a condensate since Gross-Pitaevskii equation is the equation for condensate. We conclude that our understanding of turbulence is a second order phase transition and is a condensate with molecular pairing.**

Keywords- Second Order Phase Transition; Spontaneous Symmetry Breaking; Condensate; Gross-Pitaesvkii Equation; Condensation Wave Function; Condensation Free Energy; Order Parameter; Molecular Pairing; Coupled Oscillation

I. INTRODUCTION

Turbulence is the last unsolved problem in classical physics. The major breakthrough was the work of Kolmogorov [1] in 1941 using statistical approach resulting in the concept of turbulence as an inverse cascade. N Vladimirova et al [2] also described turbulence as self-organized, i.e. the appearance of large-scale motions out of a small-scale noise. They found that an inverse cascade culminates in the creation of a spectral condensate, a mode that is spatially coherent across the whole system. In fluids, condensates are system sized vortices or zonal flows [3-5]. They found that turbulence with the condensate shares many properties with quantum systems, displaying both fluctuations and coherence. They found that the closeness with quantum system is shown most vividly within the framework of the Gross-Pitaevskii equation or nonlinear Schroedinger equation (NLSE) which is used in treating Bose Einstein condensate.

There was also the benchmark experimental work of J Nikuradze [6] in 1933 over a wide range of values of Reynolds number. Nobel Laureates like Weiner Heisenberg and Julian Schwinger had attempted to solve turbulence but unsuccessfully.

There are other works which use broken scale invariance approach on turbulence such as G Falkovitch and K R Screenivasan [7] who presented the concept of statistical conversation laws and described their role in the breaking of turbulent-state symmetries. Kolmogorov[1]'s scale-invariant theory has influenced the turbulent community for many years. However, recent experiments and calculations show departure from Kolmogorov[1]'s theory, that is, showing signs of symmetry breaking. Kolmogorov[1]'s four-fifths law exemplifies the principle of scale invariance in three-dimensional turbulence. The breakdown of scale invariance in the inertial range now called anomalous or multifractal scaling, is an important feature of turbulence. G Falkovitch et al [7] also mentioned the breakdown of scale invariance in Burgers turbulence related to an infinity of inviscid constants of motion and to the shocks that are responsible for symmetry breaking by dissipation. This description of describing scale invariance in turbulence is phenomenology. It does not explain the root and source of turbulence.

Turbulence is a very complex phenomenon and so far we are unable to explain what is happening inside turbulence. Most of the works done on turbulence are of fluid mechanics or fluid dynamics in nature. They are based on scaling argument. So far the most famous paper ever written on turbulence is that of Kolmogorov [1] which interprets turbulence as a stochastic process. More recently there are works on the small-scale structure of turbulence which shows a deep understanding of turbulence resulted from recent developments on theory and in research techniques involving increased speed and capability of computer and new measurements (over scales ranges from that of the laboratory to even that of the galaxy). Kolmogorov[1]'s 1941 paper points towards the direction of using stochastics and statistical physics in the treatment of turbulence.

In 2009 W. S.Gan [8] proposed turbulence as a second order phase transition with spontaneous symmetry breaking. Examples of second order phase transition with spontaneous symmetry breaking are magnetization, superconductivity and superfluids. This hypothesis has been subsequently supported by the experimental work of Nigel Goldenfeld [9]'s group which showed that turbulence has the same behaviour as magnetization. In his paper, he presented experimental evidence that turbulent flows are closely analogous to critical phenomena from a reanalysis of friction factor measurements in rough pipes. He found experimentally two aspects that confirm that turbulence is similar to second order phase transition such as magnetization in a ferromagnet. These are experimentally verified power law scaling of correlation functions which is reminiscent of the power-law fluctuations on many length scales that accompany critical phenomena for example in a ferromagnet near its critical point which is second order phase transition. Another aspect is the phenomena of data collapse or Widom scaling [10]. For example, in a ferromagnet, the equation of state, nominally a function of two variables, is expressible in terms of a single reduced variable which depends on a combination of external field and temperature. This has been confirmed by the experiments of J. Nikuradze [6] in 1932 and 1933which showed data collapse. Goldenfeld[9]'s work proposed that the features of the turbulence can be understood as arising from a singularity at infinite Reynolds number and zero roughness. Such singularities are known to arise in second order phase transition such as that occurs when iron is cooled down below the Curie temperature and becomes magnetic. The theory predicts that the small scale fluctuations in the fluid speed, a characteristic of turbulence are connected to the friction and this can be demonstrated by plotting the data in a special way that causes all of the Nikuradze [6] curves at different roughness to collapse into a single curve. According to Goldenfeld [9]'s study, the formation of eddies in turbulence might be a similar phenomenon to the lining of spins in magnetization. The rise of temperature in case of magnets is similar to the increase in velocity in case of the fluid passing through a pipe. Eddies are thus similar to the clusters of atoms. Goldenfeld et al [9] hope that as a result of these discoveries, the approaches that solved the problem of phase transitions will now find a new and unexpected application in providing a fundamental understanding of turbulence.

On the other hand, it seems that turbulence offers us at macroscopic level, a view of what happens during other phase transitions at microscopic level. Thus, when we are looking at eddies in a river, at turbulence, at a cascade or at cigarette smoke we can also imagine that we are actually watching what happens to molecules during a melting process or how magnetism gradually vanishes when a magnet is melted. With the mature status of statistical physics and the gauge theory in today's scientific world in this paper we have attempted to use these two available disciplines to solve turbulence.

Statistical physics was first used by Landau [11] in 1937 in the phenomenology of describing the second order phase transition, with the use of order parameter and spontaneous symmetry breaking (SSB). Examples of gauge theory are standard model of particle physics, Yang Mills theory, quantum chromodynamics and general relativity. Maxwell's equations **are** the oldest gauge theory and SSB is a property of gauge theory. The gauge theory is a more sophisticated theory than the chaos theory which is sometimes used in describing turbulence. The beauty of gauge theory is that it can be applied to both the classical regime and the quantum regime.

II. PROPOSAL OF TURBULENCE AS A SECOND ORDER PHASE TRANSITION

In 2009, W. S.Gan [8] proposed the transition of laminar flow to turbulence flow as a second order phase transition with spontaneous symmetry breaking. In this paper I extended Landau [11]'s theory of second order phase transition [11] from phenomenology to a more rigorous approach by using statistical physics and gauge theory. The Gross-Pitaevskii equation or nonlinear Schrodinger equation (NLSE) will be used. We propose turbulence as a condensate with pairing of molecules which is a property of condensate.

Our proposal that turbulence is a condensate is supported by Prof. Gregory Falkovitch's group**'s** work [2] with the following quotes from his paper: "In fluids, condensates are system sized vortices or zonal flows. Turbulence with the condensate shares many properties with quantum systems displaying both fluctuations and coherences. This closeness is shown perhaps most vividly within the framework of the nonlinear Schrodinger (Gross-Pitaevskill) equation (NSE). In their paper [2], turbulence is considered with a Gross-Pitaevskii model and the creation of a coherent condensate via an inverse cascade originating at small scales. The growth of the condensate leads to a spontaneous breaking of statistical symmetries of overcondensate fluctuations. They describe a phenomenon of spontaneous symmetry changes in the turbulent state by the change in a single parameter, the condensate level. We call this the condensate wave function in our paper. The results of their paper are supported by computer simulations. We propose that turbulence is a classical analog of the Bose Einstein condensate as the Gross-Pitaevskii equation is also used in the description of Bose Einstein condensate. Pumping to increase the condensate level is also used in Bose Einstein condensate. It has to be noted that Bose Einstein condensate such as superfluids are second order phase transiton. This in turn supports that turbulence as condensate is second phase transition.

Landau[11]'s theory of second order phase transition using order parameter is a phenomenology and a mean field theory. It has these conditions: (1) the free energy must be analytical, (2) it must be independent of the detailed mechanism that causes symmetry in the Hamiltonian. The order parameter was introduced by Landau [11] to describe spontaneous symmetry breaking. The weakness of the theory lies in its assumption of the coefficients of the order parameter and its inability to describe the fluctuations in the critical point over a range of temperatures since it is a mean field theory.

Landau [11] first showed the general relation between phase transition of the second kind and the change in the symmetry of the body. Landau [11]'s theory of second order phase transition states the thermodynamic potential or Landau free energy Φ in the format of a power series in the order parameter η:

$$\Phi(p,T,\eta) = \Phi_0 + \alpha(p,T)\eta + A(p,T)\eta^2 + B(p,T)\eta^3 + C(p,T)\eta^4 + \ \dots .... \tag{1}$$

The equilibrium state is consistent with the external parameters p and T is a minimum of the Landau free energy. This yields a prescription for determining the equilibrium value adopted by

$$\eta = \left(\frac{\partial \Phi}{\partial \eta}\right)_{\eta=\eta_{eq}} = 0 \qquad (2)$$

The Landau free energy which is equivalent to the Hamiltonian for the gauge invariance of the system (2) shows that Φ has two minima if only up to the second order is considered. The plot of Landau free energy versus the order parameter is given in Fig. 1.

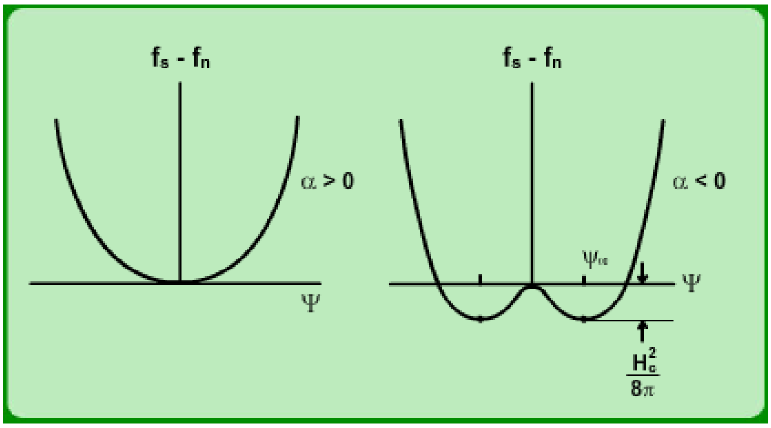

Fig. 1 Landau free energy as function of the order parameter

The left-side figure shows temperature above critical temperature and no condensation takes place. The right-side figure shows temperature at and below the critical temperature and condensation takes place and causes degeneracy in the ground state of the Landau free energy and spontaneous symmetry breaking.

Landau [11] observes that during a phase transition, the system becomes more orderly as it cools through the transition temperature. He proposed to measure the orderliness by a field parameter called the order parameter.

III. CONFIRMATION OF TURBULENCE AS A SECOND ORDER PHASE TRANSITION

There are currently two research groups in the world whose works confirmed our hypothesis that turbulence is a second order phase transition. The first is Nigel Goldenfeld's group [9] at the University of Illinois. Prof. Goldenfeld showed that the turbulent state is indeed not random but contains subtle statistical coordinations similar to those known to exist at second order phase transition such as the onset of magnetism in crystals. A metal becomes magnetic when clusters of atoms feel one another's magnetic forces and align, their magnetic moments in the same direction, like a collection of tiny arrows all adding up to one big arrow. If one heats them up, the arrows jiggle more and more and at very high temperatures, they all point in random directions. However, there is an intermediate 'Curie" temperature, where atoms can still feel one another's magnetism and form aligned clusters, although each cluster points in a random direction. According to Goldenfeld [9]'s studies the formation of eddies might be a similar phenomenon. The rise of temperature in case of magnets is similar to the increase in velocity in case of the fluid passing through a pipe. Eddies are thus similar to the cluster of atoms.

The second confirmation came from the works of the research group of Prof. Gregory Falkovitch [2] from Weizmann Institute of Science. They consider turbulence within the Gross-Pitaevskii mode and the use of this equation and look into the creation of a coherent condensate via an inverse cascade originating at small scales. The growth of the condensate leads to a spontaneous breaking of statistical symmetry of overcondensate fluctuation. This shares many properties with second order phase transition such as superconductivity.

IV. MICROSCOPIC THEORY OF TURBULENCE/TURBULENCE IS A FORM OF CONDENSATION

The phenomenon and the meaning of condensation in two-dimensional turbulence [2] is a very striking example of the self-organization of large scale coherence from small scale fluctuations. The process works via the **so-**called inverse cascade mechanism whereby energy injected into a two-dimensional fluid by a small scale stochastic force is transformed to large scales by nonlinear coupling between different scales of motion in the Navier Stokes equations. When this inverse cascade reaches the size of the system the energy carried by the inverse vascade accumulates at the largest scales and subsequently self-organizes to form large scale coherent vortices.

Condensation is self-organization. Self-organization is a process where some form of global order or coordination arises out of the local interactions between the components of an initially disordered system. This process is spontaneous, and it is not directed or controlled by any agent or subsystem inside or outside of the system. However, the laws followed by the process and its initial condition may have been chosen or caused by an agent. It is often triggered by random fluctuations that are amplified by positive feedback. The resulting organization is wholly decentralized or distributed over all the components of the system. Examples of self-organization are crystallization, magnetization, and superconductivity, where spontaneous symmetry breaking takes place.

In analogous to the role of electrons pairing in superconductor, which is responsible for superconductivity, the molecule-molecule interaction is responsible for turbulence. There are two reasons to support our hypothesis of the pairing of water molecules: one is that the Gross-Pitaevskii equation for the purpose of simplification assumes that the interactions between condensate particles are of the contact two-body type. The second reason is that to exhibit Bose Einstein condensation, the fermions must pair up to form compound particles (e.g. molecules or Cooper pairs) that are bosons. To understand turbulence, one needs to understand the detailed pairing mechanism. In turbulence, the fluid condenses and forms a ordered line up in pairs of molecules. The mechanism of pairing is a special form of nonlinear interaction analogus to the mechanism of Cooper pairing which is a special form of electron-electron interaction. Due to the attractive force of pairing the condensation free energy will go downwards to the ground state and condense and cause spontaneous symmetry breaking of the ground state of the Hamiltonian. Hence the pairing potential is the force needed to pull the pair of molecules apart. The pumping technique can be used to increase the condensate level to achieve turbulence at critical point. As condensate grows, there is symmetry breaking. The more one pumps the system the more ordered the system becomes. The pairing of molecules is a special form of molecules alignment. Pairing points towards an ordered system and the degree of pairing can be expressed in terms of entropy which can be calculated. The nonlinear interaction between molecules is responsible for pairing. The control parameter is the condensate level. The condensates are system sized vortices. An inverse cascade culminates in the creation of a spectral condensate mode that is spatially coherent and one can use the same condensate wave function throughout the space.

We expect the molecules pair to condensate until an equilibrium point is reached. As condensate grows, it leads to a spontaneous breaking of statistical symmetry. There is change in the pattern of distribution of the molecules from laminar flow to turbulence causing spontaneous symmetry breaking. Pairing is a form of molecular distribution. The condensate wave function is Landau order parameter here. These are huge number of distribution of molecular pairs and their distribution will be described by the Boltzmann distribution.

Our hypothesis that turbulence is a second order phase transiton has now been confirmed by Nigel Goldenfeld [9] and Gregory Falkovitch [2]'s research groups. Landau [9]'s theory of second order phase transition is phenomenology and a mean field theory. We will now build up a microscopic theory for turbulence. The understanding of superconductivity is that it is a Bose Einstein condensate with the mechanism which is the interaction of the electrons, the Cooper pairing of the electrons. Our classical analog for the understanding of turbulence is that it is a coherent condensate with pairing of water molecules which is the mechanism of molecular interaction. We will derive the condensation free energy for turbulence. We will show that there is spontaneous symmetry breaking at the ground sate of the condensate free energy and explain the fluctuations in the critical point over a range of temerpatures. Turbulence is well understood to be a condensate [2] and N Vladimirova et al [2]'s work also confirmed that turbulence is condensate. The Gross-Pitaevskii equation is used in their treatment as what is used for the treatment of Bose Einstein condensate.

V. DERIVATION OF THE CONDENSATION FREE ENERGY

The Gross-Pitaeskii equation for a single particle wave function is given as

$$(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^{\ 2}} + \mathrm{V}(\mathrm{r}) + \frac{4\pi\hbar^2}{M} a_s/\varphi(r)/^2)\varphi(r) = \mu\varphi(r) \tag{3}$$

where M = mass of the boson, V = external potential, a_s = boson-boson scattering length, μ = chemical potential and ħ =reduced Planck,*s* constant.

The Landau order parameter here is the condensate wave function. The Gross-Pitaevskii equation describes the ground state of a quantum system of identical bosons using the Hartree-Fock approximation and the potential interaction model.

In the Hartree-Fock approximation, the total wave function ψ of the system of N bosons is taken as a product of single-particle function ψ:

$$\Psi(\overrightarrow{r_1},\overrightarrow{r_2},......................,\overrightarrow{r_N}) = \psi(\overrightarrow{r_1})\psi(\overrightarrow{r_2})..........................\psi(\overrightarrow{r_N})$$

The pseudopotential model Hamiltonian of the system is given as

$$\mathrm{H} = \sum_{i=1}^{N}(-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial r_i^2} + \mathrm{V}(\overrightarrow{r_i})) + \sum_{i<j}\frac{4\pi\hbar^2}{M} a_s\delta(\overrightarrow{r_i}-\overrightarrow{r_j}) \tag{4}$$

where $\delta(\vec{r})$ is the Dirac delta-function. (3) is the model equation for the single particle wave function in a Bose-Einstein condensate. It is similar in form to the Ginzburg-Landau equation. Since turbulence is condensate, the Gross-Pitaevskii equation is used here. This equation has also been used by N. Vladimirova et al [2] in their description of turbulence.

A Bose-Einstein condensate is a gas of bosons that are in the same quantum state and thus can be described by the same wave function. A free quantum particle is described by a single-particle Schrodinger equation. Interaction between particles in a real gas is taken into account by pertinent many-body Schrondinger equation. If the average spacing between the particles in a gas is greater than the scattering length, then one can approximate the true interaction potential that features in this equation by a pseudopotential. The nonlinearity of the Gross-Pitaesvskii equation has its origin in the interaction between the particles. This becomes evident by equating the coupling constant of interaction in the Gross-Pitaevskii equation with zero on which the single-particle Schrondinger equation describing a particle inside a trapping potential is recovered.

Then the equation has the form of the Schrondinger equation with the addition of an interaction term. The coupling constant g is proportional to the scattering length a_s of two interacting bosons:

$$g=\frac{4\pi\hbar^2}{M} \quad a_s \tag{5}$$

The energy density is

$$\varepsilon = \frac{\hbar^2}{2M} /\nabla\Psi(\overrightarrow{r})/^2 + V(\vec{r})/\Psi(\overrightarrow{r})/^2 + \frac{1}{2}g/\Psi(\overrightarrow{r})/^4 \tag{6}$$

where Ψ = condensation wave function = Landau order parameter and V = external potential.

The condensation free energy = ΔF = is given by the difference between the turbulence and laminar states at temperature T [12]. From (6), the turbulence state free energy is given by

$$\varepsilon_t = \frac{\hbar^2}{2M} /\nabla\Psi(\overrightarrow{r})/^2 + V(\overrightarrow{r})/\Psi(\overrightarrow{r})/^2 + \frac{1}{2} g_t/\Psi(\overrightarrow{r})/^4 \tag{7}$$

and $\frac{1}{2} g_t/\Psi(\overrightarrow{r})/^4$ = nonlinear interaction term responsible for interaction.

We consider that turbulence is due to the nonlinear interaction between the molecules and the mechanism of this interaction is the pairing of molecules, a special case of the nonlinear interaction.

For laminar state, from (6), the free energy is given by

$$\varepsilon_l = \frac{\hbar^2}{2M} /\nabla\Psi(\overrightarrow{r})/^2 + V(\overrightarrow{r})/\Psi(\overrightarrow{r})/^2 + \frac{1}{2} g_l/\Psi(\overrightarrow{r})/^4 \tag{8}$$

Therefore

$$\Delta\varepsilon = \varepsilon_l - \varepsilon_t = /\Psi(\overrightarrow{r})/^2 \, (V_l(\overrightarrow{r}) - V_t(\vec{r})) + \frac{1}{2}/\Psi(\overrightarrow{r})/^4 (g_l - g_t) \tag{9}$$

This will be the ground state of the Hamiltonian of turbulence free energy

The critical value of the order parameter is corresponding to that for minimum condensation free energy and this is when turbulence occurs and can be obtained as follows:

$$\frac{\partial\Delta\varepsilon}{\partial\Psi(\overrightarrow{r})} = 2(V_l(\overrightarrow{r}) - V_t(\vec{r}))\Psi(\overrightarrow{r}) + 2(g_l - g_t)/\Psi(\overrightarrow{r})/^3 = 0 \tag{10}$$

This will produce $\Psi(\overrightarrow{r}) = 0$ and

$$2\,(g_l - g_t)/\Psi(\overrightarrow{r})/^2 + 2(V_l - V_t) = 0$$

$$/\Psi(\overrightarrow{r})/^2 = -2\,(V_l - V_t)/2(g_l - g_t)$$

or

$$\Psi(\overrightarrow{r}) = \pm\sqrt{\frac{V_{t-V_l}}{g_{l-g_t}}} \tag{11}$$

The meaning of $\Psi(\overrightarrow{r}) = 0$ means that there is only one value for the order parameter and there is no broken symmetry. (11) shows that there are two values of the order parameter or condensation wave function when turbulence occurs or when there is a degeneracy of the ground state of the Hamiltonian. This confirms spontaneous symmetry breaking in the ground state of condensation free energy when turbulence occurs. The forms of the plot of the condensation free energy versus the order parameter is the same as that given in Fig. 1.

Now condensation free energy for the critical value of $\Psi(\overrightarrow{r})$ when turbulence occurs **is** given from (9) and (11) by:

$$\Delta\varepsilon_{condensation} = \frac{1}{2}[V_l(\overrightarrow{r}) - V_t(\overrightarrow{r})]^2/(g_t - g_l) \tag{12}$$

VI. THE NAMBU GOLDSTONE MODE IN TURBULENCE

In magnets, the original rotational symmetry (present in the absence of an external magnetic field) is spontaneously broken so that the magnetization points into a specific direction. The Nambu Goldstone bosons are then the magnons, i.e. spin waves in which the local magnetization direction oscillates.

For turbulence, as a source of sound [13], the Nambu Goldstone mode is the phonon. Phonon induces the pairing of molecules resulting in spontaneous symmetry breaking of the ground state of the condensation free energy.

VII. NUMERICAL CALCULATIONS OF THE CRITICAL VALUE OF THE CONDENSATION FREE ENERGY

The critical value of the condensation free energy is when turbulence occurs and is given by (12). To evaluate the numerical value, one would need to know the numerical values of V_l,$V_{t,}g_{l,}$ and g_t.

For laminar flow, there is no molecular pairing and from (5), the coupling constant,g_l = $\frac{4\pi\hbar^2}{M_l}a_{sl}$ where h = Planck's constant, a_{sl} = scattering length = radius of water molecule for elastic scattering case and M_l = mass of a water molecule.

To calculate the radius of a water molecule, we assume that the two hydrogen atoms and the oxygen atom are in close compact form and all along the same row. Therefore, radius of a water molecule = $\frac{1}{2}(d_1 + d_1 + d_2)$ where d_1 = diameter of hydrogen atom and d_2 = diameter of oxygen.

This gives radius of a water molecule = (2x5.292 + 2x$\frac{4.8}{2}$) x 10^{-11}m = 1.5384 x 10^{-10}m = a_s = scattering length.

The mass of a water molecule = 2xmass of hydrogen atom + mass of oxygen atom = 2x 1.6759x 10^{-24} gm + 2.67804x10^{-23}gm = 3.01322x10^{-23}gm.

Hence g_l = $\frac{4\pi(1.054571726x10^{-34})^2}{3.01322x10^{-23} \quad sec^2x10^{-3kg}}m^4kg^2x15.384x10^{-11}$m= 7.1359x$10^{-52}m^5kgsec^{-2}$

For turbulence flow, the coupling constant = g_t= $\frac{4\pi\hbar^2 a_{st}}{M_t}$. Since turbulence is a condensate, there is pairing of two water molecules. We shall use the coupled oscillators model (Fig. 2) of classical mechanics to describe the pairing. We assume that the two water molecules are coupled together with a distance l in between. Then the radius of the coupled water molecules will be$\frac{1}{2}$($4d_1$ + $2d_2$) + l/2 = $2d_1$ + d_2 + l/2 = 3.0768x 10^{-10}m + $l/2$.

Scattering length = radius of coupled molecules = a_{st} = 2xa_{sl} + $l/2$

Then g_t = $4\pi\hbar^2 a_{st}/M_t$ = g_l + $\frac{\pi\hbar^2 l}{M_l}$ = 7.1359 x $10^{-52}m^5$kgsec^{-2} + 3.6436 x$10^{-42}lm^5$kgsec^{-2}

Now V_l = external potential for the laminar flow case. For the laminar flow case, we shall assume that a single water molecule consists of a line of two hydrogen atoms and one oxygen atom. Then the external potential is the potential energy put into the string connecting the atoms by stretching = $\rho c^2\varepsilon^2$L/2 where L = length of string. Assuming that the atoms are in close compact, then L = distance between the centre of the hydrogen atom at one end and the centre of the oxygen atom at the other end, ε = small fractional increase of L due to stretching of string, c = speed of propagation of particle vibration mode = sound velocity in pure water = 1,440 m/sec. Here we represent a line of atoms by a continuous string with certain average properties, such as the mass per unit length. Hence ρ = 1/d = mass of water molecule/L.

Assume that the two hydrogen atoms and oxygen atom are in line and in close contact,

$\rho = 3.01322\ x\ 10^{-23} gm$/(4x5.292 + 2x4.8)x$10^{-11}$m= 9.79335x$10^{-17}$kg/m

L = d_1/2 + d_1 + d_2/2 = 2.0676x $10^{-10}m$

Then V_l =ε^2 9.79335x10^{-17}kg/m x1,440$^2m^2sec^{-2}$x2.0676x10^{-10}m/^{2}= ε^2x 2.0993882x10^{-20}kgm^2sec^{-2} (13)

For turbulence flow, with the coupled oscillator model (Fig. 2), the length of string equals from the centre of the hydrogen atom from one end to the centre of the oxygen atom at the other end, assuming that the atoms of the two pairing water molecules have a small space of l between them.

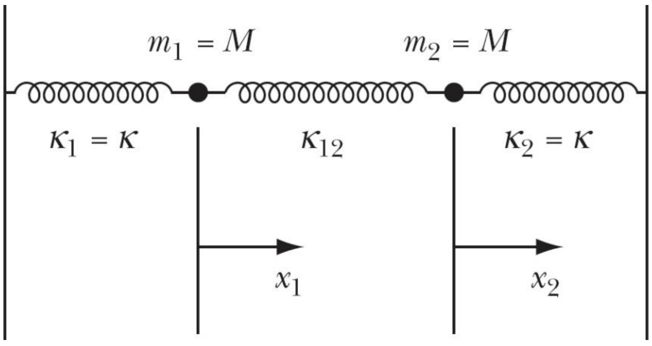

Fig. 2 The coupled oscillator model (M = mass of oscillator;κ = spring constant; x= displacement)

_____• __________ • ________________ • _______

A B C

Case of laminar flow

One water molecule, the hydrogen atoms and oxygen atom are assumed to be tightly compact in the molecule. A,B,C are centres of the hydrogen atoms and the oxygen atom respectively.

d_1 = diameter of the hydrogen atom and $d_2 =$ diameter of the oxygen atom

AB = d_1, BC = $(d_1 + d_2)/2$

_____• ___________•_________________•_________……..._____•___________•_________________•_________

A_1 B_1 C_1 l A_2 B_2 C_2

Case of turbulence flow

Two water molecules in pairing are represented by the coupled oscillator model.$A_1.B_1$,C_1 = centres of hydrogen atoms and oxygen atom of the first water molecule respectively.A_2,B_2,C_2 = centres of hydrogen atoms and oxygen atoms of the second water molecule respectively. l = distance between the two water molecules.

$$A_1B_1 = A_2B_2 = d_1,\ B_1C_1 = B_2C_2 = (d_1 + d_2)/2$$

Fig. 3 Lining of water molecules in the laminar flow and turbulence cases

Then L = $\frac{7}{2}d_1 + \frac{3}{2}d_2 + l$ = (7x5.292x10^{-11} + 3x4.8x10^{-11} + l)m= (5.1444 + l)x10^{-10}m

M = mass per unit length = mass of water molecule/diameter of water molecule

= (2x0.167x$10^{-23} + 2.678x10^{-23})gm/(d_1 + d_1 + d_2)$cm

= 9.78939x 10^{-17}kgm^{-1}

Then $V_t = \frac{1}{6}B^2Mc^2L^3$

= $\frac{1}{6}B^2x0.78939x10^{-17}kgm^{-1}$x $1{,}440^2m^2sec^2x(5.1444 + l)^3x10^{-33}m^3$

= B^2(4.6063x$10^{-39} + 3.38325x10^{-44}l^3$)kg$m^4sec^{-2}$ (14)

Now condensation free energy for the critical value of $\psi(\overrightarrow{r})$ when turbulence occurs **is** given by substituting (11) into (9), as follows

$$\Delta\varepsilon = \frac{1}{2}\frac{[V_l(\overrightarrow{r}) - v_t(\overrightarrow{r})]^2}{(g_{t-g_l})} \quad (15)$$

$$=\frac{\frac{1}{2}[\varepsilon^2x2.0994x10^{-20}kgm^2sec^{-2} - B^2(4.6063x10^{-39}+3.38325x10^{-44}l^{3)}kgm^4sec^{-2}]^2}{3.6436x10^{-42}lm^5kgsec^{-2}} \quad (16)$$

To calculate the numerical value for l:

The diameter of the hydrogen molecule = $4r_1$+ $2r_2$ = 4x5.292x10^{-11}m + 2x4.8x10^{-11}m = 3.0768x10^{-10}m.

We choose $l = \frac{1}{4}$x 3.0768x10^{-10}m = 7.692 x 10^{-11}m.

Substituting in expression for g_t, we obtain g_t = 8.02799 x $10^{-52}m^5$kgsec^{-2}.

To calculate the numerical value forε^2:

We choose ε = 5%, then ε^2 = 2.5 x 10^{-3}.

Substituting in (13), we have

$$V_l = 5.2485 \text{ x } 10^{-23} \text{ kg}m^2sec^{-2}$$

To calculate the numerical value for B^2 :

B = small fractional increase in the y direction of the string of the coupled oscillator and has a quadratic dependence on the increase in the x direction. Hence B = $\varepsilon^2 =$ 2.5x10^{-3} and $B^2 =$ 6.25x10^{-6}.

Substituting in (14), we have

$$V_t = 2.914375\ x\ 10^{-44}\text{kg}m^4sec^{-2} + 9.623328 \text{ x } 10^{-80}\text{kg}m^4sec^{-2}$$

In order to be in conformity with the unit in V_l, we will choose the M in V_t to be mass per unit volume.

The critical value of condensation wave function is given by

$\Psi(\overrightarrow{r}) = \pm\sqrt{\frac{V_t(\overrightarrow{r})-V_l(\overrightarrow{r})}{g_{l-}g_t}}$

$= \pm\sqrt{\frac{(2.914375x10^{-44}-5.2485x10^{-23})kgm^4sec^{-2}}{(7.1359x10^{-52}-8.02799x10^{-52})m^5kgsec^{-2}}}$

$= \pm\sqrt{\frac{5.2485x10^{-23}}{0.89209x10^{-52}}}m^{-0.5}$

$= \pm\sqrt{5.8833749x10^{19}}m^{-0.5}$

$= \pm 7.670315\text{x}10^9 m^{-0.5}$

From (15), the condensation free energy is given by $\frac{1}{2}/\Psi(\overrightarrow{r})/^2$ x (V_l - V_t) = $\frac{1}{2}$ x $(\pm 7.670315x10^9)^2m^{-1}$x(5.2485x$10^{-23}$)$kgm^2sec^{-2}$. = 1.5439442x$10^{-3}$$kgmsec^{-2}$=1.5439442 gmm$sec^{-2}$

VIII. ANALYSIS OF THE EQUATIONS FOR THE EXTERNAL POTENTIAL FOR THE LAMINAR FLOW AND THE TURBULENCE FLOW

For the laminar flow, the external potential or potential energy is given by $V_l(\overrightarrow{r}) = \frac{1}{2}\varepsilon^2\rho c^2$L whereas for the turbulence flow, it is given by $V_t(\boldsymbol{\vec{r}}) = \frac{1}{6}B^2\text{m}c^2L^3$. When examining the form of the two equations, they are of the same form except that for the laminar flow there is a linear dependence on the length of the oscillating string whereas for the turbulence flow due to the effect of the pairing or coupling of water molecules there is a nonlinear dependence on the length of the spring which is reasonable.

Further, our formula for the condensation free energy agrees in form with the formula from the Landau [11] phenomenology. The first portion of the Ginszburg-Landau free energy density is the Landau term:

$$\varepsilon = \propto [\Psi(\overrightarrow{r})]^2 + \frac{\beta}{2}[\Psi(\overrightarrow{r})]^4 + \ldots\ldots\ldots\ldots\ldots \quad (17)$$

where $\Psi(\overrightarrow{r})$ is the order parameter which is the condensation wave function for our case when applied to turbulence. The expression for our condensation free energy derived using the Gross-Pitaevskii equation has the form:

$$\Delta\varepsilon = \varepsilon_l - \varepsilon_t = [V_l(\overrightarrow{r}) - V_t(\overrightarrow{r})] \ /\psi(\overrightarrow{r})/^2 + \frac{1}{2}(g_l\text{-}g_t)/\psi(\overrightarrow{r})/^4$$

which is of the same form as Landau free energy when expanded in power series of the order parameter with:

$$\alpha = V_l(\overrightarrow{r}) - V_t(\vec{r}) \text{ and } \frac{\beta}{2} = \frac{1}{2}(g_l\text{- } g_t) \quad (18)$$

This confirms that turbulence is a second order phase transition as it gives the same formula as Landau free energy for second order phase transition. It also confirms that turbulence is a condensate as the Gross-Pitaevskii equation which is meant for condensate is used for our derivation.

IX. ANALYSIS OF THE PLAUSITY OF THE NUMERICAL VALUES OF THE COUPLING CONSTANT AND CONDENSATION FREE ENERGY

Our numerical value for the coupling constant for the turbulence case is almost ten orders of magnitude larger than that for the laminar case. This is reasonable during turbulence there is an additional coupling from the pairing of the water molecules.

The expression for the condensation free energy is given by:

$$\Delta\varepsilon_{condensation} = \frac{1}{2}[V_l(\overrightarrow{r}) - V_t(\overrightarrow{r})]^2/(g_t - g_l)$$

Since $g_t > g_{l,}$this value is always positive which is plausible.

X. CONCLUSIONS

We show that turbulence is a second order phase transition with spontaneous symmetry breaking. We also understand turbulence as a condensate with pairing of water molecules. It is of interest that we use the Gross-Pitaevskii equation, the equation for condensate and used by Bose Einstein condensate and superconductivity, to derive the condensation free energy and obtain the same form of equation as that used in Landau phenomenology, that is, as a power series in the order parameter, the condensation wave function. This further confirms that turbulence is a condensate.

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