

Wave Approach for Calculation of Natural Frequencies and Mode Shapes in a Waveguides with Discrete Damping

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Abstract- In many practical applications, continuous systems may interact with discrete damping elements. For example, discrete damping elements are routinely attached to structures for vibration control. In this article, wave propagation method is applied for the calculation of frequencies and mode shapes for one-dimensional waveguides with a discrete damping.

Keywords- Wave Propagation; Reflection; Damping; Natural Frequencies; Mode Shapes

I. INTRODUCTION

Vibration of uniform waveguides, like rods and beams, has been studied with different approaches, such as the approximate, the analytical, and the wave methods. Nikkhah-Bahrami [1] used a wave approach to calculate frequencies & mode shapes of rods and beams, of which analytical solution is available. Nikkhah-Bahrami et al [1] and Longhmani [2] used a wave approach to analyse the non-uniform 1D waveguides of which analytical solution is available, such as the polynomial or the exponential cross-section. Also khoshbayani and Nikkhah- Bahrami [3] used the modified wave approach for the calculation of natural frequencies and mode shapes in arbitrary non-uniform 1D waveguides.

In all of the above studies, the conventional boundary condition is used. In this article, the wave propagation method is applied for the calculation of frequencies and mode shapes of 1D waveguides with a discrete damping.

II. METHODOLOGY FOR THE CALCULATION OF NATURAL FREQUENCIES AND MODE SHAPES OF ROD

The equation of motion for a rod is as follows:

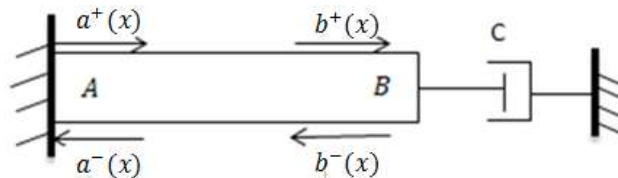


Fig. 1 Wave propagation in the rod

$$E \frac{\partial^2 u(x, t)}{\partial x^2} = \rho \frac{\partial^2 u(x, t)}{\partial t^2} \quad (1)$$

where E , ρ and $u(x, t)$ are elasticity modulus, density and displacement along the rod respectively.

Assume:

$$u(x, t) = u(x)g(t)$$

then,

$$u(x, t) = (E_1 e^{-i\beta x} + E_2 e^{i\beta x}) e^{i\omega t} \quad (2)$$

where,

$$\beta = \omega \sqrt{\frac{\rho}{E}}$$

The term $E_1 e^{-i\beta x}$ is for positive moving wave and the term $E_2 e^{i\beta x}$ is for negative moving wave.

With the assumptions below:

$$\begin{aligned} a^+ &= E_1 e^{-i\beta x} \\ a^- &= E_2 e^{i\beta x} \end{aligned} \quad (3)$$

We have,

$$u(x, t) = (a^+ + a^-)e^{i\omega t} \quad (4)$$

Boundary condition for $x = L$ with the presence of the damper is considered as below:

$$EA \frac{\partial u(x, t)}{\partial x} = -C \frac{\partial u(x, t)}{\partial t} \quad (5)$$

where A and C are area cross section and damping ratio respectively.

$$\begin{aligned} \frac{\partial u}{\partial x} &= i\beta(-b^+(L) + b^-(L))e^{i\omega t} \\ \frac{\partial u}{\partial t} &= i\omega(b^+(L) + b^-(L))e^{i\omega t} \end{aligned}$$

Substituting $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial t}$ in Eq. (4) leads to equation

$$\begin{aligned} EAi\beta(-b^+(L) + b^-(L))e^{i\omega t} &= i\omega(b^+(L) + b^-(L))e^{i\omega t} \cdot C \\ b^+(L) \frac{C\omega + EA\beta}{-C\omega + EA\beta} &= b^-(L) \end{aligned} \quad (6)$$

As a result, reflection number of the damper boundary is as below:

$$\begin{aligned} r_B \cdot b^+(L) &= b^-(L) \\ r_B &= \frac{C\omega + EA\beta}{-C\omega + EA\beta} \end{aligned} \quad (7)$$

Boundary condition for $x = 0$ with the presence of the spring is considered as below:

$$EA \frac{\partial u(x, t)}{\partial x} = -K u(x, t) \quad (8)$$

where K is the spring stiffness.

With Substituting $\frac{\partial u(x, t)}{\partial x}$ and $u(x, t)$ in Eq. (4) leads to equation

$$\begin{aligned} EAi\beta(-a^+(0) + a^-(0))e^{i\omega t} &= -K(a^+(0) + a^-(0)) \\ a^+(0) &= \frac{K + EA\beta i}{-K + EA\beta i} a^-(0) \end{aligned} \quad (9)$$

As a result, reflection number of the springer boundary is as below:

$$\begin{aligned} a^+(0) &= r_A \cdot a^-(0) \\ r_A &= \frac{K + EA\beta i}{-K + EA\beta i} \end{aligned} \quad (10)$$

When the spring stiffness coefficient goes to infinity, the support is in the form of clamped-clamped and the reflection number on the boundary is equal to -1. In result

$$r_A = -1 \quad (11)$$

By calculating transmission number, we have [4]:

$$b^+(x) = a^+(x + L) = F^+(L)a^+(x) \quad (12)$$

$$a^-(x) = b^-(x + L) = F^-(L)b^-(x) \quad (13)$$

After Substituting Eq. (12) in Eq. (4), the result leads to

$$F^+(L) = F^-(L) = e^{-i\beta L} \quad (14)$$

Satisfying the boundary conditions will yield the characteristic equation for wave numbers by which the wave numbers can be found. Thereby, using the relationship between the natural frequencies and wave number, the natural frequencies are calculated.

For Fig. 1 with writing Eqs. (7), (10), (12) and (13) in matrix form, we have

$$\begin{bmatrix} -1 & r_A & 0 & 0 \\ f(L) & 0 & -1 & 0 \\ 0 & -1 & 0 & f(L) \\ 0 & 0 & r_B & -1 \end{bmatrix} \begin{Bmatrix} b^+ \\ b^- \\ a^+ \\ a^- \end{Bmatrix} = 0 \quad (15)$$

This equation has non-trivial solution, if the determinant of coefficients equals zero and leads to

$$r_A r_B f(L)^2 - 1 = 0 \quad (16)$$

Substituting Eqs. (7), (11), (12) and (13) into Eq. (16), leads to:

$$r_A r_B e^{-2i\beta L} - 1 = 0 \quad (17)$$

or

$$e^{-2i\beta L} = \frac{C - \sqrt{\rho EA}}{C + \sqrt{\rho EA}}$$

Applying Ln function on both sides of Eq. (17)

$$\begin{aligned} -2i\beta L &= \ln \left(\frac{C - \sqrt{\rho EA}}{C + \sqrt{\rho EA}} \right) \\ i\beta L &= -\frac{1}{2} \ln \left(\frac{C - \sqrt{\rho EA}}{C + \sqrt{\rho EA}} \right) \end{aligned} \quad (18)$$

$$i\beta L = -\frac{1}{2} \ln \left(\frac{C - \sqrt{\rho EA}}{C + \sqrt{\rho EA}} \right) = \frac{-1}{2} \ln \left| \frac{C - \sqrt{\rho EA}}{C + \sqrt{\rho EA}} \right| - \frac{1}{2} i \arg \left(\frac{C - \sqrt{\rho EA}}{C + \sqrt{\rho EA}} \right)$$

For preferred angle, two conditions are possible

A. The term $\frac{C - \sqrt{\rho EA}}{C + \sqrt{\rho EA}}$ is equal to a real positive value

B. The term $\frac{C - \sqrt{\rho EA}}{C + \sqrt{\rho EA}}$ is equal to a real negative value

In the first condition, angle value is equal to $\pm 2k\pi$ in the second condition this angle is equal to $\pm(2k + 1)\pi$. Result to below equation:

$$\begin{aligned} i\beta L &= \frac{-1}{2} \ln \left| \frac{C - \sqrt{\rho EA}}{C + \sqrt{\rho EA}} \right| \pm \frac{1}{2} i(2k\pi) \quad C > \sqrt{\rho EA} \\ i\beta L &= \frac{-1}{2} \ln \left| \frac{C - \sqrt{\rho EA}}{C + \sqrt{\rho EA}} \right| \pm \frac{1}{2} i(2k + 1)\pi \quad C < \sqrt{\rho EA} \end{aligned} \quad (19)$$

Substituting $\beta = \omega \sqrt{\frac{\rho}{E}}$, the equations are simplified as:

$$\begin{aligned} \omega &= \frac{1}{2L} \sqrt{\frac{E}{\rho}} \left(i \ln \left| \frac{C + \sqrt{\rho EA}}{C - \sqrt{\rho EA}} \right| \pm (2k\pi) \right) \quad C > \sqrt{\rho EA} \\ \omega &= \frac{1}{2L} \sqrt{\frac{E}{\rho}} \left(i \ln \left| \frac{C + \sqrt{\rho EA}}{C - \sqrt{\rho EA}} \right| \pm (2k + 1)\pi \right) \quad C < \sqrt{\rho EA} \end{aligned} \quad (20)$$

Now, we can assume that $\omega = a + ib$

b is the imaginary part of the eigenvalue ω and represents the damped natural frequency of the damped system, while a is the real part of ω and is equal to the decay ratio $\zeta \omega_n$ of the free vibration amplitude with ζ denoting damping ratio corresponding to undamped natural frequency ω_n of the system.

According to Eq. (20), when there is no damping in the boundary, the value of c is zero and the free clamped condition is obtained and reflection matrix on the boundary is equal to 1 and natural frequency from Eq. (20) is obtained as

$$\omega_n = \pm \frac{1}{2L} \sqrt{\frac{E}{\rho}} (2k + 1)\pi \quad (21)$$

When the damping coefficient goes to infinity, the support is in the form of clamped-clamped and the reflection matrix on the boundary equal to -1, and the natural frequency of the system is described as

$$\omega_n = \pm \frac{k\pi}{L} \sqrt{\frac{E}{\rho}} \quad (22)$$

When $C = \sqrt{\rho EA}$, according to Eq. (20), the solution goes to infinity and consequently there is no eigenvalue. The reason is that when $C = \sqrt{\rho EA}$, the wave striking to boundary doesn't reflect. On the other hand the reflection number equals to zero which is obvious from Eq. (7). Thus, the results are exactly coincident with analytical solution [1].

III. ANALYTICAL METHOD FOR THE CALCULATION OF NATURAL FREQUENCIES

Motion equation of a bar is written as:

$$\frac{E}{\rho} \frac{d^2 u(x, t)}{dx^2} = \frac{d^2 u(x, t)}{dt^2} \quad (23)$$

Using variables separation method, the equation is solved as below:

$$u(x, t) = U(x) \cdot g(t) \quad (24)$$

Substituting Eq. (24) into Eq. (23):

$$\frac{E}{\rho} \frac{1}{U(x)} \frac{d^2 U(x)}{dx^2} = \frac{1}{g(t)} \frac{d^2 g(t)}{dt^2} = \lambda \quad (25)$$

Equalling this value to Eigen value λ , two equation systems are obtained and solving them results $U(x)$ & $g(t)$:

$$\frac{d^2 g(t)}{dt^2} - \lambda g(t) = 0 \Rightarrow g(t) = a_1 e^{\sqrt{\lambda}t} + a_2 e^{-\sqrt{\lambda}t} \quad (26)$$

It is necessary to limit the time to infinitive in order to obtain limited displacement for achieving system stability. To reach that, the term $a_1 e^{\sqrt{\lambda}t}$ needs to be zero as well as equalling time function values to:

$$g(t) = a_2 e^{-\sqrt{\lambda}t} \quad (27)$$

Solving the equation of displacement function:

$$\frac{d^2 U(x)}{dx^2} - \frac{\lambda \rho}{E} U(x) = 0 \Rightarrow U(x) = b_1 e^{\sqrt{\frac{\rho \lambda}{E}}x} + b_2 e^{-\sqrt{\frac{\rho \lambda}{E}}x} \quad (28)$$

Imposing boundary conditions, b_1 and b_2 are obtained:

$$U(0) = 0 \Rightarrow b_1 + b_2 = 0 \quad (29)$$

Boundary condition for $x = L$ in the presence of a damper is:

$$EA \frac{du(x, t)}{dx} = -C \frac{\partial u(x, t)}{\partial t} \quad (30)$$

Substituting $\frac{du(x, t)}{dx}$ and $\frac{\partial u(x, t)}{\partial t}$ in Eq. (23):

$$EA \left[\sqrt{\frac{\rho \lambda}{E}} e^{\sqrt{\frac{\rho \lambda}{E}}x} + \sqrt{\frac{\rho \lambda}{E}} e^{-\sqrt{\frac{\rho \lambda}{E}}x} \right] g(t) = -C(-\sqrt{\lambda})g(t)U(L) \quad (31)$$

Eliminating $g(t)$ from two sides of equations and substituting $U(x)$ in equation:

$$\begin{aligned} EA \left[\sqrt{\frac{\rho \lambda}{E}} e^{\sqrt{\frac{\rho \lambda}{E}}L} + \sqrt{\frac{\rho \lambda}{E}} e^{-\sqrt{\frac{\rho \lambda}{E}}L} \right] &= C \sqrt{\lambda} [e^{\sqrt{\frac{\rho \lambda}{E}}L} - e^{-\sqrt{\frac{\rho \lambda}{E}}L}] \\ \Rightarrow [EA \sqrt{\frac{\rho}{E}} - C] e^{\sqrt{\frac{\rho \lambda}{E}}L} + [EA \sqrt{\frac{\rho}{E}} - C] e^{-\sqrt{\frac{\rho \lambda}{E}}L} &= 0 \end{aligned} \quad (32)$$

Factoring logarithmic term of equation:

$$[EA \sqrt{\frac{\rho}{E}} - C] e^{2\sqrt{\frac{\rho \lambda}{E}}L} + [EA \sqrt{\frac{\rho}{E}} + C] = 0 \quad (33)$$

$$\Rightarrow e^{2\sqrt{\frac{\rho\lambda}{E}}L} = \frac{\left[EA\sqrt{\frac{\rho}{E}} + C\right]}{\left[-EA\sqrt{\frac{\rho}{E}} + C\right]} = \frac{C + A\sqrt{\rho E}}{C - A\sqrt{\rho E}}$$

It is obtained from Eq. (33):

$$e^{2\sqrt{\frac{\rho\lambda}{E}}L} = \frac{C + A\sqrt{\rho E}}{C - A\sqrt{\rho E}} \quad (34)$$

The left side logarithmic term can be written in two ways:

$$\begin{aligned} e^{2\sqrt{\frac{\rho\lambda}{E}}L} &= e^{2\sqrt{\frac{\rho\lambda}{E}}L \pm 2k\pi i} & C > \sqrt{\rho EA} \\ e^{2\sqrt{\frac{\rho\lambda}{E}}L} &= e^{2\sqrt{\frac{\rho\lambda}{E}}L \pm (2k+1)\pi i} & C < \sqrt{\rho EA} \end{aligned} \quad (35)$$

Because

$$\begin{aligned} e^{2k\pi i} &= \cos[2k\pi] + i\sin[2k\pi] = 1 \\ e^{(2k+1)\pi} &= \cos[(2k+1)\pi] + i\sin[(2k+1)\pi] = -1 \end{aligned} \quad (36)$$

So:

$$\begin{cases} e^{2\sqrt{\frac{\rho\lambda}{E}}L \pm 2k\pi i} = \frac{C + A\sqrt{\rho E}}{C - A\sqrt{\rho E}} & C > \sqrt{\rho EA} \\ e^{2\sqrt{\frac{\rho\lambda}{E}}L \pm (2k+1)\pi i} = \frac{C + A\sqrt{\rho E}}{C - A\sqrt{\rho E}} & C < \sqrt{\rho EA} \end{cases} \quad (37)$$

Imposing Ln function to both sides of Eq. (37):

$$\begin{cases} 2\sqrt{\frac{\rho\lambda}{E}}L \pm 2k\pi i = \ln \left[\frac{C + A\sqrt{\rho E}}{C - A\sqrt{\rho E}} \right] & C > \sqrt{\rho EA} \\ 2\sqrt{\frac{\rho\lambda}{E}}L \pm (2k+1)\pi i = \ln \left[\frac{C + A\sqrt{\rho E}}{C - A\sqrt{\rho E}} \right] & C < \sqrt{\rho EA} \end{cases} \quad (38)$$

As a result:

$$\begin{cases} \sqrt{\lambda} = \frac{1}{2L} \sqrt{\frac{E}{\rho}} \left(\ln \left[\frac{C + A\sqrt{\rho E}}{C - A\sqrt{\rho E}} \right] \pm 2k\pi i \right) & C > \sqrt{\rho EA} \\ \sqrt{\lambda} = \frac{1}{2L} \sqrt{\frac{E}{\rho}} \left(\ln \left[\frac{C + A\sqrt{\rho E}}{C - A\sqrt{\rho E}} \right] \pm (2k+1)\pi i \right) & C < \sqrt{\rho EA} \end{cases} \quad (39)$$

As it is seen, the results obtained from both wave propagation technique and Analytical method (Eq. (20) & Eq. (39)) are in a good agreement with each other. Considering $\lambda = -\omega^2$ system natural frequencies can be calculated.

IV. METHODOLOGY FOR THE CALCULATION OF MODE SHAPES OF ROD

In order to calculate mode shapes, it is sufficient that the moving positive and negative waves are summed together.

$$\begin{aligned} \phi(x) &= a^+ + a^- \\ a^+ &= E_1 e^{-i\beta x} \\ a^- &= E_2 e^{i\beta x} \end{aligned} \quad (40)$$

Substituting the value of obtained natural frequency in Eq. (20), the mode shapes are calculated.

$$\phi(x) = E_1 e^{-i\omega \sqrt{\frac{\rho}{E}}x} + E_2 e^{i\omega \sqrt{\frac{\rho}{E}}x} \quad (41)$$

Substituting the value of ω into eq.41, it leads to one complex number for each $\phi(x)$. The real part of this number equals to vibration amplitude, and its imaginary part is equal to phase difference of various moving points. In this situation, there is no possible way to demonstrate the mode shapes without damper. Generally, system coordinate doesn't move in the same phase or out of phase any more. Therefore, phase concept is used. The behaviour of different parts of rod can be deduced from phase plots. The phase plots for two states $C > \sqrt{\rho EA}$ and $C < \sqrt{\rho EA}$ are shown in Figs. 2, 3, 4 and 5.

State 1: Phase plot for $C > \sqrt{\rho EA}$

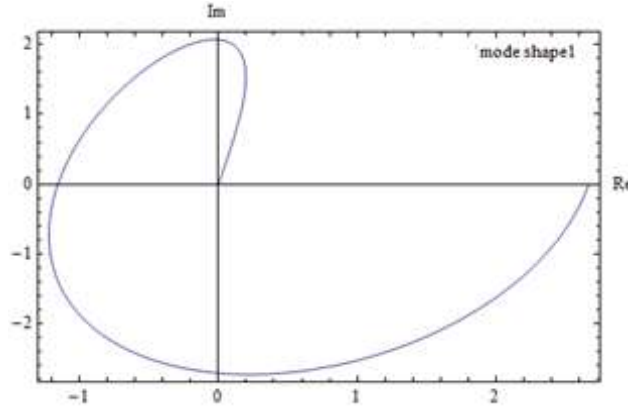


Fig. 2 Phase plot for first mode shape for $C > \sqrt{\rho EA}$

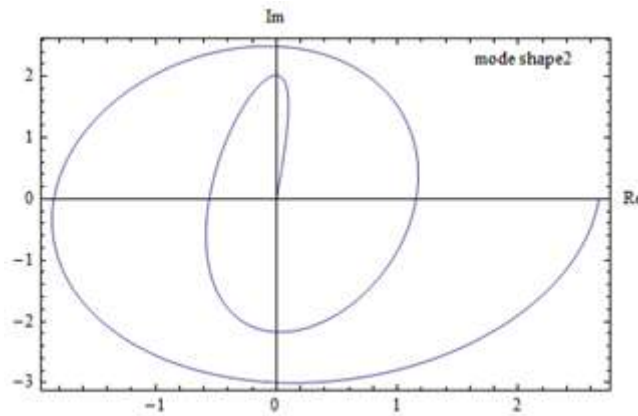


Fig. 3 Phase plot for second mode shape for $C > \sqrt{\rho EA}$

State2: Phase plot for $C < \sqrt{\rho EA}$

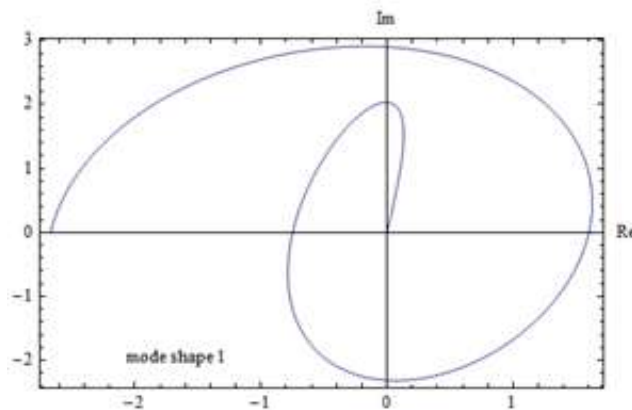


Fig. 4 Phase plot for first mode shape for $C < \sqrt{\rho EA}$

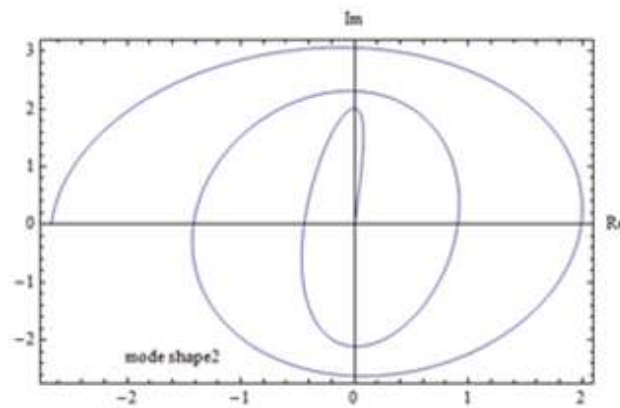


Fig. 5 Phase plot for second mode shape for $C < \sqrt{\rho EA}$

The point (0, 0) of the phase diagram corresponds the left rod boundary condition. The end point corresponds to the rod which the damper is connected.

CONCLUSION

The paper shows the capability of wave approach for handling frequencies & mode shapes of wave guides with non-conventional boundary conditions.

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