Application of the $(\frac{G'}{G})$-expansion Method to Generalized Kawahara Equation and Generalized KdV Equation

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***Abstract*-In this paper, we apply $(\frac{G'}{G})$-expansion method to obtain the travelling wave solutions of the generalized Kawahara equation and the generalized KdV equation. We use the hyperbolic functions and trigonometric functions and rational functions to express the exact solutions of the generalized Kawahara equation and the generalized KdV equation.**

***Keywords*- $(\frac{G'}{G})$-Expansion Method; Generalized Kawahara Equation; Generalized Kdv Equation**

I. INTRODUCTION

In the past four decades, numerous people are greatly concerned with the exact solutions of nonlinear partial differential equations resulting from mathematical physics, fluid mechanics, plasma physics, optical fibers, solid state physics, chemical physics and geochemistry. Lots of effective and powerful methods have been established by many researchers, such as the homogeneous balance method [1, 2], inverse scattering method [3], the tanh-function expansion and its various extension [4, 5], the homotopy perturbation method [6], the sine-cosine method [7] and Exp-function method [8] as well as Lie symmetry analysis method [9-12].

In [13], the authors firstly proposed the new method called the $(\frac{G'}{G})$-expansion method to seek the travelling solutions of KdV equation and mKdV equation as well as Hirota-Satsuma equations. It has been widely applied to derive the nonlinear transformations and exact solutions of NLEEs [14-24]. The $(\frac{G'}{G})$-expansion method is based on the explicit linearization of NLEEs for traveling waves with a certain substitution which leads to a second-order differential equation with constant coefficients. Moreover, it transforms a nonlinear equation to a simple algebraic computation,we can obtain more general solutions with some free parameters by the method. If we set the parameters in the obtained wider set of solutions as special values, then some previously known solutions can be recovered.

In this paper, we use $(\frac{G'}{G})$-expansion method to look for the travelling wave solutions of the generalized Kawahara equation and the generalized KdV equation which arise in mathematical physics [25, 26].

II. SUMMARY OF THE $(\frac{G'}{G})$-EXPANSION METHOD

In the following we would like to outline the main steps of the $(\frac{G'}{G})$-expansion method [13, 15]:

Step 1. For a given nonlinear partial differential equation (NPDE) system

$$F(u,u_t,u_x,u_{tt},u_{xt},u_{xx}...)=0, \tag{2.1}$$

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can be converted to on ODE

$$P(\varphi, -v\varphi', \varphi', v^2\varphi'', -v\varphi'', \varphi''...) = 0, \tag{2.2}$$

using a traveling wave variable $u(x,t) = \varphi(\xi), \xi = x - Vt$.

Step 2. If the solution of the ODE can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$u(\xi) = \sum_{i=0}^{n} a_i (\frac{G'}{G})^i, \tag{2.3}$$

where $G = G(\xi)$ satisfies the following second order linear constant differential equation.

$$G'' + \lambda G' + \mu G = 0, \tag{2.4}$$

where λ, μ are constants and $G' = \frac{dG}{d\xi}, G'' = \frac{d^2G}{d\xi^2}, \xi = x - vt$, *v* is a constant.

Where $a_i (i = 0,1,2...n)$, λ and μ are constants to be determined later, $a_n \neq 0$, the unwritten part in (2.3) is also a polynomial in $(\frac{G'}{G})$, but the degree of which is generally equal to or less than *n*-1, the positive integer n can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in ODE (2.2).

Step 3. By substituting (2.3) into Eq. (2.2) and using second order LCDE (2.4), collecting all terms with the same order of $(\frac{G'}{G})$ together, the left-hand side of Eq. (2.2) is converted into another polynomial in $(\frac{G'}{G})$. Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $a_i (i = 0,1,2...n)$, V, λ and μ.

Step 4. Supposed that the constants $a_i (i = 0,1,2...n)$, V, λ and μ can be obtained by solving the algebraic equations in Step 3, since the general solutions of the second order LCDE (2.4) have been well known for us, then substituting $a_1, a_2, a_3....a_n, V$ and the general solutions of Eq. (2.4) into (2.3) , we have more traveling wave solutions of the nonlinear partial differential equation (2.1).

III. APPLICATION TO THE GENERALIZED KAWAHARA EQUATION

All paragraphs must be indented. All paragraphs must be justified, *i.e.* both left-justified and right-justified. In this section, we consider the generalized Kawahara equation in the following form

$$u_t + pu_{xxxxx} + qu_{xxx} + \frac{1}{5}\partial_x(u^5) = 0, \tag{3.1}$$

Where *p* is nonzero real constant and $q \in R$ denote the highest order dispersive coefficient and the lowest order dispersive coefficient, respectively [27]. (3.1) is called the singularly perturbed Korteweg-de Vries equation to describe the evolution of solitary waves of small amplitude in the case where the Bond number is less than but close to $\frac{1}{3}$ and the Froude number is close to 1 [28, 29]. We introduce a travelling wave variable

$$u(\xi) = u(x,t), \xi = x - vt, \tag{3.2}$$

where the speed *v* of the travelling waves is to be determined later. Substituting (3.2) into (3.1) yields

$$-vu' + pu''''' + qu''' + \frac{1}{5}(u^5)' = 0, \tag{3.3}$$

where the prime denotes the derivation with the respect to the variable ξ. Integrating it with respect to the variable ξ leads to

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$$c - vu + pu'''' + qu'' + \frac{1}{5}u^5 = 0, \tag{3.4}$$

Where c is an integration constant that is to be determined later. By using the idea of $(\frac{G'}{G})$-expansion method [13], we balance the terms u'''' and u^5, then we obtain $5m = m+4$ which yields $m=1$. Thus according to $(\frac{G'}{G})$-expansion method [13], we formally write the solution

$$u(\xi) = a_0 + a_1(\frac{G'}{G}), \tag{3.5}$$

where a_0, a_1 are to be determined. To solve (3.4), we give the following expressions

$$u'' = a_1\left\{2(\frac{G'}{G})^3 + 3\lambda(\frac{G'}{G})^2 + (2\mu + \lambda^2)\frac{G'}{G} + \lambda\mu\right\} \tag{3.6}$$

$$\begin{aligned} u'''' &= 24a_1(\frac{G'}{G})^5 + 60\lambda a_1(\frac{G'}{G})^4 + (60\mu + 50\lambda^2)a_1(\frac{G'}{G})^3 \\ &(60\mu\lambda + 15\lambda^3)a_1(\frac{G'}{G})^2 + (16\mu^2 + 22\lambda^2\mu + \lambda^4)a_1(\frac{G'}{G}) \\ &+(8\lambda\mu^2 + \mu\lambda^3)a_1 \end{aligned} \tag{3.7}$$

$$\begin{aligned} u^5 &= a_0^5 + 5a_0^4a_1(\frac{G'}{G}) + 10a_0^3a_1^2(\frac{G'}{G})^2 + 10a_0^2a_1^3(\frac{G'}{G})^3 \\ &+5a_0a_1^4(\frac{G'}{G})^4 + a_1^5(\frac{G'}{G})^5 \end{aligned} \tag{3.8}$$

Substituting (3.5)-(3.8) into (3.4) yields a polynomial. Equating each coefficient of this polynomial to zero leads to a set of algebraic equations for a_0, a_1, C, V as follows:

$$(\frac{G'}{G})^5 : \frac{1}{5}a_1^5 + 24a_1p = 0 \tag{3.9}$$

$$(\frac{G'}{G})^4 : 60pa_1\lambda + a_0a_1^4 = 0 \tag{3.10}$$

$$(\frac{G'}{G})^3 : pa_1(40\mu + 50\lambda^2) + 2a_1q + 2a_0^2a_1^3 = 0 \tag{3.11}$$

$$(\frac{G'}{G})^2 : pa_1(60\lambda\mu + 15\lambda^3) + 3a_1\lambda q + 2a_0^3a_1^2 = 0 \tag{3.12}$$

$$(\frac{G'}{G}) : -Va_1 + pa_1(16\mu^2 + 22\lambda^2\mu + \lambda^4) + (2\mu + \lambda^2\mu_1)qa_1 + a_0^4a_1 = 0 \tag{3.13}$$

$$(\frac{G'}{G})^0 : C - Va_0 + pa_1(8\lambda\mu^2 - \lambda^3\mu) + \mu\lambda qa_1 + \frac{1}{5}a_0^5 = 0 \tag{3.14}$$

Solving (3.9)-(3.14) by Matlab yields the following two sets of solutions.

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Case 1.

$$a_0 = -\frac{60p\lambda}{(-120p)^{\frac{3}{4}}} \tag{3.15}$$

$$a_1 = (-120p)^{\frac{1}{4}} \tag{3.16}$$

$$V = (16\mu^2 + 22\lambda^2\mu + \lambda^4)p + (2\mu + \lambda^2)q + \frac{(60p\lambda)^4}{(-120p)^3} \tag{3.17}$$

$$\begin{aligned} C = &-(16\mu^2 + 22\lambda^2\mu + \lambda^4)\frac{60p^2\lambda}{(-120p)^{\frac{3}{4}}} - (2\mu + \lambda^2)\frac{60p^2\lambda}{(-120p)^{\frac{3}{4}}} \\ &-(8\lambda\mu^2 + \lambda^3\mu)(-120p)^{\frac{1}{4}} + \mu\lambda q(-120p)^{\frac{1}{4}} - \frac{4}{5}\frac{(60p\lambda)^5}{(-120p)^{\frac{15}{4}}}t \end{aligned} \tag{3.18}$$

where

$$\lambda^2 - 4\mu = \frac{q}{5p} \tag{3.19}$$

Case 2.

$$a_0 = \frac{60p\lambda}{(-120p)^{\frac{3}{4}}} \tag{3.20}$$

$$a_1 = -(-120p)^{\frac{1}{4}} \tag{3.21}$$

$$V = (16\mu^2 + 22\lambda^2\mu + \lambda^4)p + (2\mu + \lambda^2)q + \frac{(60p\lambda)^4}{(-120p)^3} \tag{3.22}$$

$$\begin{aligned} C = &(16\mu^2 + 22\lambda^2\mu + \lambda^4)\frac{60p^2\lambda}{(-120p)^{\frac{3}{4}}} + (2\mu + \lambda^2)\frac{60p^2\lambda}{(-120p)^{\frac{3}{4}}} \\ &+(8\lambda\mu^2 + \lambda^3\mu)(-120p)^{\frac{1}{4}} + \mu\lambda q(-120p)^{\frac{1}{4}} + \frac{4}{5}\frac{(60p\lambda)^5}{(-120p)^{\frac{15}{4}}}t \end{aligned} \tag{3.23}$$

Where

$$\lambda^2 - 4\mu = \frac{q}{5p} \tag{3.24}$$

Substituting (3.15)-(3.18) into (3.5) yields

$$u(\xi) = -\frac{60p\lambda}{(-120p)^{\frac{3}{4}}} + (-120p)^{\frac{1}{4}}(\frac{G'}{G}) \tag{3.25}$$

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Where

$$\xi = x - \left\{(16\mu^2 + 22\lambda^2\mu + \lambda^4)p + (2\mu + \lambda^2)q + \frac{(60p\lambda)^4}{(-120p)^3}\right\}t$$

Substituting (3.20)-(3.23) into (3.5) yields

$$u(\xi) = \frac{60p\lambda}{(-120p)^{\frac{3}{4}}} - (-120p)^{\frac{1}{4}}\left(\frac{G'}{G}\right) \tag{3.26}$$

where

$$\xi = x - \left\{(16\mu^2 + 22\lambda^2\mu + \lambda^4)p + (2\mu + \lambda^2)q + \frac{(60p\lambda)^4}{(-120p)^3}\right\}t.$$

Substituting the general solution of (3.4) into (3.25) (3.26), we obtain two types of travelling wave equations of generalized Kawahara equation as follows.

When $\lambda^2 - 4\mu > 0$,

$$u_1 = (-120p)^{\frac{1}{4}}\frac{\sqrt[2]{\lambda^2 - 4\mu}A}{2} \text{ and } u_2 = -(-120p)^{\frac{1}{4}}\frac{\sqrt[2]{\lambda^2 - 4\mu}A}{2} \tag{3.27}$$

With $A = \dfrac{C_1 \sinh\frac{1}{2}\sqrt[2]{\lambda^2 - 4\mu}\xi + C_2 \cosh\frac{1}{2}\sqrt[2]{\lambda^2 - 4\mu}\xi}{C_1 \cosh\frac{1}{2}\sqrt[2]{\lambda^2 - 4\mu}\xi + C_2 \sinh\frac{1}{2}\sqrt[2]{\lambda^2 - 4\mu}\xi}$

where

$$\xi = x - \left\{(16\mu^2 + 12\lambda^2\mu + \lambda^4)p + (2\mu + \lambda^2)q + \frac{(60p\lambda)^4}{(-120p)^3}\right\}t$$

and $c_j(1 \le j \le 2)$ are arbitrary constants.

When $\lambda^2 - 4\mu < 0$

$$u_3 = (-120p)^{\frac{1}{4}}\frac{\sqrt[2]{4\mu - \lambda^2}B}{2} \text{ and } u_4 = -(-120p)^{\frac{1}{4}}\frac{\sqrt[2]{4\mu - \lambda^2}B}{2} \tag{3.28}$$

with

$$B = \frac{-C_1 \sin\frac{1}{2}\sqrt[2]{4\mu - \lambda^2}\xi + C_2 \cos\frac{1}{2}\sqrt[2]{4\mu - \lambda^2}\xi}{C_1 \cos\frac{1}{2}\sqrt[2]{4\mu - \lambda^2}\xi + C_2 \sin\frac{1}{2}\sqrt[2]{4\mu - \lambda^2}\xi}$$

where

$$\xi = x - \left\{(16u^2 + 12\lambda^2 u + \lambda^4)p + (2\mu + \lambda^2)q + \frac{(60p\lambda)^4}{(-120p)^3}\right\}t$$

and $c_j(1 \le j \le 2)$ are arbitrary constants.

When $\lambda^2 - 4\mu = 0$ that is q=0 we have

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$$u_5 = (-120p)^{1/4}\frac{C_2}{C_1 + C_2\xi} \text{ and } u_6 = -(-120p)^{1/4}\frac{C_2}{C_1 + C_2\xi},$$

Where $\xi = x - (6q\mu - 40p\mu^2)t$ and $c_j(1 \le j \le 2)$ are arbitrary constants.

IV. APPLICATION TO THE GENERALIZED KDV EQUATION

In this section, we consider the generalized KdV equation in the following form

$$u_t + u_{xxx} + 2uu_{xx} + 2(u_x)^2 + \frac{1}{2}\partial x(u^2) = 0 \tag{4.1}$$

Equation (4.1) is proposed in [25, 26]. Equation (4.1) is KdV hierarchy, it describes the models in water wave problems and in elastic media. We introduce a travelling wave variable

$$u(\xi) = u(x,t), \quad \xi = x - Vt \tag{4.2}$$

where the speed *V* of the travelling waves is to be determined later. Substituting (4.2) into (4.1) yields

$$-Vu' + u''' + (u^2)'' + \frac{1}{2}(u^2)' = 0 \tag{4.3}$$

where the prime denotes the derivation with the respect to the variable ξ. Integrating it with respect to the variable ξ leads to

$$C - Vu + u'' + (u^2)' + \frac{1}{2}u^2 = 0 \tag{4.4}$$

where *C* is an integration constant that is to be determined later. By using the idea of $(\frac{G'}{G})$ -expansion method [13], we balance the terms u'' and $(u^2)'$, then we obtain 2*m*+1 = *m*+2 which yields *m* =1. Thus according to $(\frac{G'}{G})$-expansion method [13], we formally write the solution

$$u(\xi) = a_0 + a_1(\frac{G'}{G}) \tag{4.5}$$

where a_0, a_1 are to be determined. We assume that $a_1 \neq 0$ since u= a_0 is obviously the solution of (4.1), where a_0 is arbitrary constant.

To solve (4.4), we give the following expressions

$$u' = a_1\left\{-(\frac{G'}{G})^2 - \lambda(\frac{G'}{G}) - \mu\right\} \tag{4.6}$$

$$u'' = a_1\left\{2(\frac{G'}{G})^3 + 3\lambda(\frac{G'}{G})^2 + (2\mu + \lambda^2)\frac{G'}{G} + \lambda\mu\right\} \tag{4.7}$$

$$u_2 = a_0^2 - 2a_0a_1(\frac{G'}{G}) + a_1^2(\frac{G'}{G})^2 \tag{4.8}$$

Substituting (4.5)-(4.8) into (4.4) yields a polynomial in $(\frac{G'}{G})$.

Equating each coefficient of this polynomial to zero leads to a set of algebraic equations for a_0, a_1, C, V as follows:

$$(\frac{G'}{G})^3 : 2a_1 - a_1^2 = 0 \tag{4.9}$$

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$$(\frac{G^{'}}{G})^2 : 3\lambda a_1 - 2a_0 a_1 - 2a_1^2 \lambda + \frac{a_1^2}{2} = 0 \tag{4.10}$$

$$(\frac{G^{'}}{G}) : -Va_1 + a_1(2\mu + \lambda^2) - 2\lambda a_0 a_1 - 2\lambda a_1^2 + a_0 a_1 = 0 \tag{4.11}$$

$$(\frac{G^{'}}{G})^0 : C - Va_0 + a_1 \lambda \mu - 2\mu a_0 a_1 + \frac{a_0^2}{2} = 0 \tag{4.12}$$

Solving (4.9)-(4.12) by Matlab yields

$$a_0 = \frac{\lambda}{2} + \frac{1}{4} \tag{4.13}$$

$$a_1 = 1 \tag{4.14}$$

$$V = \frac{1}{4} \tag{4.15}$$

$$C = -\frac{\lambda^2}{8} + \frac{\mu}{2} + \frac{1}{32} \tag{4.16}$$

Substituting (4.13)-(4.16) into (4.4) yields

$$u(\xi) = \frac{1}{2} + \frac{1}{4} + \frac{G^{'}}{G} \tag{4.17}$$

where

$$\xi = x - \frac{1}{4}t$$

Substituting the general solution of (4.4) into (4.5),

We obtain three types of travelling wave equations of generalized KdV equation as follows.

When $\lambda^2 - 4\mu > 0$,

$$u_1 = \frac{1}{4} + \frac{\sqrt[2]{\lambda^2 - 4\mu}\mathrm{A}}{2} \tag{4.18}$$

with

$$A = \frac{C_1 \sinh \frac{1}{2} \sqrt[2]{\lambda^2 - 4\mu}\xi + C_2 \cosh \frac{1}{2} \sqrt[2]{\lambda^2 - 4\mu}\xi}{C_1 \cosh \frac{1}{2} \sqrt[2]{\lambda^2 - 4\mu}\xi + C_2 \sinh \frac{1}{2} \sqrt[2]{\lambda^2 - 4\mu}\xi}$$

Where $\xi = x - \frac{1}{4}t$ and $c_j (1 \le j \le 2)$ are arbitrary constants.

When $\lambda^2 - 4\mu < 0$,

$$u_2 = \frac{1}{4} + \frac{\sqrt[2]{4\mu - \lambda^2}\mathrm{B}}{2} \tag{4.19}$$

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with

$$B = \frac{-C_1 \sin\frac{1}{2}\sqrt[2]{4\mu-\lambda^2}\xi + C_2\cos\frac{1}{2}\sqrt[2]{4\mu-\lambda^2}\xi}{C_1\cos\frac{1}{2}\sqrt[2]{4\mu-\lambda^2}\xi + C_2\sin\frac{1}{2}\sqrt[2]{4\mu-\lambda^2}\xi}$$

Where $\xi = x - \frac{1}{4}t$ and $c_j (1 \le j \le 2)$ are arbitrary constants.

When $\lambda^2 - 4\mu = 0$,

$$u_3 = \frac{1}{4} + \frac{c_2}{c_1 + c_2\xi} \tag{4.20}$$

Where $\xi = x - \frac{1}{4}t$ and $c_j (1 \le j \le 2)$ are arbitrary constants.

V. SUMMARY AND CONCLUSIONS

We use $(\frac{G'}{G})$-expansion method to successfully obtain the exact solution of the generalized Kawahara and the generalized KdV equation. The procedure of finding solutions is simple and standard [13]. The solutions that we obtain can be expressed by hyperbolic functions, trigonometric functions and the rational functions. The process of finding exact solutions of generalized Kawahara equation demonstrates that $(\frac{G'}{G})$-expansion method is an effective tool for solving higher order dispersive equations in mathematical physics.

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