On the Estimation of the Vertical Gradient of Normal Gravity on the Earth's Physical Surface

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***Abstract*-The vertical derivative of normal gravity at a specific point *P* is given by the Bruns formula. The Bruns formula contains the mean curvature of a chosen normal equipotential surface. To use the Bruns formula for a chosen point above the ellipsoid, the mean curvature of the normal equipotential surface passing through this point needs to be known. In this work a simple approach is presented to deal with this problem, by deriving a formula to express the mean curvature of the normal equipotential surfaces using three variables: geodetic latitude, geodetic longitude and geometric height. With this formula which is valid for points on the Earth's surface, a more general form of the Bruns formula has been constructed which allows the determination of the vertical derivative of normal gravity. The general form of the Bruns formula can be extended for any point above the reference ellipsoid of revolution.**

Keywords- Normal Gravity Field / Equipotential Surfaces / Normal Gravity Vector / Vertical Gradient

I. INTRODUCTION

An approximation of the Earth's gravity field can be obtained by an ellipsoidal model field. For our purpose the model field is generated by an ellipsoid of revolution. The surface of this ellipsoid is considered as an equipotential surface. The ellipsoid of revolution contains the mass of the Earth and its center of mass coincides with the center of a rotating global Cartesian coordinate system (X, Y, Z). The Z-axis is along the axis of rotation of the ellipsoid, the X-axis is the intersection of the zero meridian plane and the equator's plane and the Y-axis completes the right-handed system. The potential which is generated by the ellipsoid is called normal potential denoted by U. The normal gravity field was studied by Pizzetti and Somigliana [1], [2]. The normal potential (or normal gravity potential) is given by the following formula

$$U(u,\beta)=\frac{GM}{E}\tan^{-1}\frac{E}{u}+\frac{1}{2}\omega^2a^2\frac{q}{q_0}(\sin^2\beta-\frac{1}{3})+\frac{1}{2}\omega^2(u^2+E^2)\cos^2\beta \tag{0}$$

where G is the gravitational constant, M is the mass of the Earth, ω is the mean angular velocity of the Earth and (u, β, λ) are the ellipsoidal coordinates which are related to the Cartesian coordinates with the following relations

$$\begin{aligned} X&=\sqrt{u^2+E^2}\cos\beta\cos\lambda\\ Y&=\sqrt{u^2+E^2}\cos\beta\sin\lambda\\ Z&=u\sin\beta \end{aligned} \tag{0a}$$

In addition

$$E^2=a^2-b^2 \tag{0b}$$

$$q=q(u)=\frac{1}{2}\left[\left(1+3\frac{u^2}{E^2}\right)\arctan\left(\frac{E}{u}\right)-3\frac{u}{E}\right] \tag{0c}$$

$$q_0=q(b) \tag{0d}$$

where a and b stand for the semimajor and semiminor axis of the ellipsoid respectively.

The Bruns formula – for the Earth's normal gravity field – expresses the vertical gradient of normal gravity at an arbitrary point P in a local rotating Cartesian system (x, y, z) whose origin is the point P. The x – axis is tangent to the normal equipotential

surface pointing east, the y – axis is also tangent to the normal equipotential surface pointing north, and z – axis is vertical to the equipotential surface i.e. vertical to the tangent plane of the normal equipotential surface at point P. The Bruns formula is given by the following relation [3]

$$\left.\frac{\partial \gamma}{\partial z}\right|_P = -2\omega^2 - 2\gamma_P J_P \tag{0e}$$

where γ_P stands for the magnitude of the normal gravity vector at point P and J_P stands for the mean curvature of the normal equipotential surface at the same point. The vertical gradient of normal gravity can be used for the determination of the vertical gradient of the actual gravity. The vertical gradient of actual gravity is essential for the connection between absolute and relative gravity networks [4], [5]. The value of the vertical gradient of gravity is taken approximately as 3.086 μGal/m. Now available gravimetric technologies provide gravity with accuracy that makes use of this value of normal vertical gradient for gravity measurement reductions unsatisfying. Studies concerning the determination of actual vertical gravity gradient are still open in terms of methods and instruments used. For example, the value of the actual vertical gradient of gravity can be evaluated directly with the use of gravimeters [6]. An alternative method is to split the value of the actual gravity gradient into a normal part (value of the normal vertical gradient) and a disturbing part [3]. The normal vertical gradient is significant not only for surface gravity measurements and gravity networks. The normal vertical gradient is also a part of what we call normal gravitational tensor [7]. The components of the normal gravitational tensor are essential in satellite gradiometry [8], and airborne gradiometry, amongst others, hence the normal vertical gradient is of great importance.

Hirvonen [9] in 1960 presented the following formula for the vertical gradient of normal gravity

$$\left.\frac{\partial \gamma}{\partial z}\right|_P = -\frac{GM}{a^2 b}(2 + e^2 - m - 3e^2 \sin^2 \phi_P + 5m \sin^2 \phi_p) + \frac{3GM}{a^3 b} h_P \tag{0f}$$

where e is the first eccentricity of the ellipsoid of revolution and

$$m = \frac{\omega^2 a^2 b}{GM} \tag{0g}$$

Formula (0f) contains only a linear term for the description of the variation of the vertical gradient of normal gravity above the surface of the ellipsoid. Later Tscherning [10] in 1976 provided a program calculating normal gravity field quantities. Amongst others, Šprlák [11] in 2012 provided an interface for the normal gravitational tensor components, making use of series expansions in spherical coordinates which can be used at high altitudes. Our formula will be expressed as a function of (φ, λ, h) coordinates i.e. geodetic latitude, geodetic longitude, and geometric height. The geodetic coordinates are given from GPS. The quantities which we use to construct our formula are very informative for the geometry of the normal equipotential surfaces i.e. it is possible to compare geometric quantities of a normal equipotential surface with those of the ellipsoid of revolution. We mention that if P is a point above the ellipsoid of revolution and ε is the vertical line to the ellipsoid passing through point P then the geodetic latitude of point P is the angle between the line ε and the equatorial plane of the ellipsoid. The meridian plane of point P is the plane which contains the line ε and the Z – axis. The geodetic longitude of point P is the angle between the XZ plane and the meridian plane of point P. The geometric height of point P is the length of the line ε between point P and the intersection point Q of the line ε and the ellipsoid of revolution. The geodetic coordinates are related with the global Cartesian coordinates with the following relations [3]

$$\begin{aligned} X &= (R_2 + h)\cos\phi\cos\lambda \\ Y &= (R_2 + h)\cos\phi\sin\lambda \\ Z &= \left(\frac{b^2}{a^2} R_2 + h\right)\sin\phi \end{aligned} \tag{0h}$$

where R_2 is the radius of curvature of the ellipsoid of revolution in the direction of the prime vertical. The formula for the aforementioned curvature will be given in the next paragraph.

The derivation of our improved formula for the normal vertical gradient consists of four steps: a) Expression of the fundamental quantities E, F, G, L, M, N of the ellipsoid of revolution in geodetic coordinates, b) formulation of the differential equation of the generator curve of the normal equipotential surface passing through point P (this formulation is necessary for the expression of the first and second order partial derivatives of geometric height in geodetic coordinates), c) formulation of the fundamental quantities

E_e, F_e, G_e, L_e, M_e, N_e of the normal equipotential surface passing through point P in geodetic coordinates, and d) expressing the principal curvatures of the normal equipotential surface passing through point P and its mean curvature in geodetic coordinates. We will describe the generator curve in the next section. Since the mean curvature of the normal equipotential surface passing through point P will contain the geometric height it is possible to determine the mean curvature of any equipotential surface at a specific point with known geometric height. Finally, we will formulate a relation similar to the original Bruns formula which will describe the vertical derivative of normal gravity above the surface of the ellipsoid.

II. METHODOLOGY

Step a): Relations for the fundamental quantities of the ellipsoid of revolution in geodetic coordinates.

Let the vector equation of the ellipsoid of revolution in geodetic coordinates be:

$$\bar{s}(\phi,\lambda) = (R_2 \cos\phi \cos\lambda, R_2 \cos\phi \sin\lambda, \frac{b^2}{a^2} R_2 \sin\phi) \tag{1}$$

where

$$R_1 = \frac{1}{k_1} = \frac{a^2}{b} \frac{1}{(1+e'^2 \cos^2\phi)^{3/2}} \tag{2}$$

$$R_2 = \frac{1}{k_2} = \frac{a^2}{b} \frac{1}{(1+e'^2 \cos^2\phi)^{1/2}} \tag{3}$$

are the principal radii of curvature of the ellipsoid of revolution.

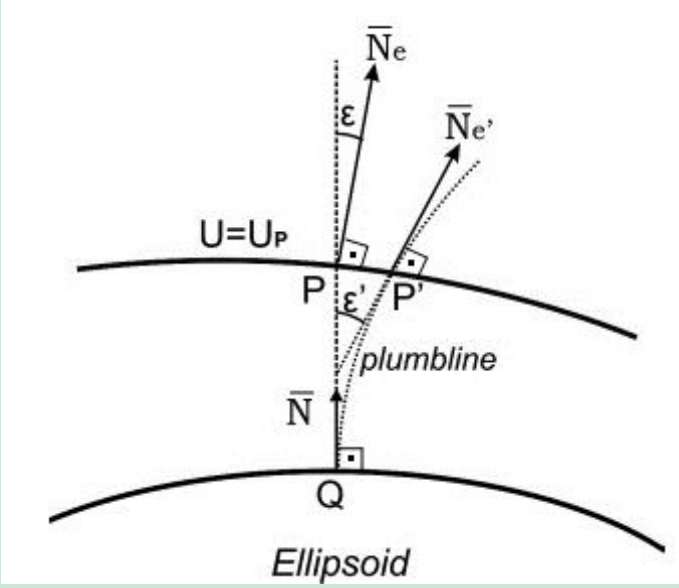

Fig. 1 Equipotential surface at point P and the ellipsoid

Now let a point P above the ellipsoid (see Fig. 1) with known coordinates (φ_P, λ_P, h_P) and the equipotential surface passing through this point with normal potential U_P. Since the distance PP' is very small, we set $\varepsilon' = \varepsilon$. Then the vector equation of the normal equipotential surface passing through point P can be written as

$$\bar{s}_e(\phi,\lambda) = \bar{s}(\phi,\lambda) + h(\phi)\bar{N}(\phi,\lambda) \tag{4}$$

where $\bar{N}$ stands for the unit normal vector of the aforementioned equipotential surface. The geometric height is a function of the geodetic latitude when it is confined along the normal equipotential surface. We have the following relations

$$\frac{\partial \overline{s}_e}{\partial \phi} = \frac{\partial \overline{s}}{\partial \phi} + \frac{dh}{d\phi}\overline{N} + h\frac{\partial \overline{N}}{\partial \phi} \tag{5}$$

$$\frac{\partial \overline{s}_e}{\partial \lambda} = \frac{\partial \overline{s}}{\partial \lambda} + h\frac{\partial \overline{N}}{\partial \lambda} \tag{6}$$

The unit normal vector of the equipotential surface passing through point P is given by the following formula

$$\overline{N}_e = \frac{1}{\left|\frac{\partial \overline{s}_e}{\partial \phi} \times \frac{\partial \overline{s}_e}{\partial \lambda}\right|}\left(\frac{\partial \overline{s}_e}{\partial \phi} \times \frac{\partial \overline{s}_e}{\partial \lambda}\right) \tag{7}$$

Since we confine ourselves on the Earth's physical surface we put ε = ε′. The unit normal vector of the equipotential surface can also be written as

$$\overline{N}_e = \overline{N}\cos\varepsilon + \frac{1}{\left|\frac{\partial \overline{s}}{\partial \phi}\right|}\frac{\partial \overline{s}}{\partial \phi}\sin\varepsilon \tag{8}$$

where

$$\overline{N} = \frac{1}{\left|\frac{\partial \overline{s}}{\partial \phi} \times \frac{\partial \overline{s}}{\partial \lambda}\right|}\left(\frac{\partial \overline{s}}{\partial \phi} \times \frac{\partial \overline{s}}{\partial \lambda}\right) \tag{9}$$

is the unit normal vector of the ellipsoid of revolution and [3]

$$\varepsilon' = h\frac{f^*}{R}\sin 2\phi \tag{10}$$

where R = 6371008 m, f^* = 0.00511 and ε is the angle between the two aforementioned unit vectors. For our calculations we will need the fundamental quantities of the ellipsoid of revolution. These are equal to

$$E = \langle \frac{\partial \overline{s}}{\partial \phi}, \frac{\partial \overline{s}}{\partial \phi} \rangle = \left[R_2{}^2 + \frac{b^4}{a^4}\left(\frac{\partial R_2}{\partial \phi}\right)^2\right]\sin^2\phi + \left[\left(\frac{\partial R_2}{\partial \phi}\right)^2 + \frac{b^4}{a^4}R_2{}^2\right]\cos^2\phi + \\ + \left(\frac{b^2}{a^2} - 1\right)\frac{\partial R_2}{\partial \phi}R_2\sin 2\phi \tag{11}$$

$$F = \langle \frac{\partial \overline{s}}{\partial \phi}, \frac{\partial \overline{s}}{\partial \lambda} \rangle = 0 \tag{12}$$

$$G = \langle \frac{\partial \overline{s}}{\partial \lambda}, \frac{\partial \overline{s}}{\partial \lambda} \rangle = R_2{}^2\cos^2\phi \tag{13}$$

$$L = Ek_1 = \left\{ \left[R_2{}^2 + \frac{b^4}{a^4}\left(\frac{\partial R_2}{\partial \phi}\right)^2 \right] \sin^2\phi + \left[\left(\frac{\partial R_2}{\partial \phi}\right)^2 + \frac{b^4}{a^4} R_2{}^2 \right] \cos^2\phi + \right.$$
$$\left. + \left(\frac{b^2}{a^2} - 1\right) \frac{\partial R_2}{\partial \phi} R_2 \sin 2\phi \right\} \frac{b}{a^2} (1 + e'^2 \cos^2\phi)^{3/2} \tag{14}$$

$$M = \langle \overline{N}, \frac{\partial^2 \overline{s}}{\partial \phi \partial \lambda} \rangle = 0 \tag{15}$$

$$N = Gk_2 = \frac{a^2}{b} \frac{\cos^2\phi}{(1 + e'^2 \cos^2\phi)^{1/2}} \tag{16}$$

Step b): Determination of the differential equation of the generator curve of the normal equipotential surface passing through point *P*

The generator curve of the normal equipotential surface passing through point *P* is a plane curve on the meridian plane. If we rotate this curve along the *Z* – axis we will form the normal equipotential surface passing through point *P* (normal equipotential surfaces are rotational surfaces). We start with the equation of the vertical vector of the normal equipotential surface (see relation (7)).

$$\frac{\partial \overline{s}_e}{\partial \phi} \times \frac{\partial \overline{s}_e}{\partial \lambda} = \left(\frac{\partial \overline{s}}{\partial \phi} + \frac{dh}{d\phi} \overline{N} + h \frac{\partial \overline{N}}{\partial \phi} \right) \times \left(\frac{\partial \overline{s}}{\partial \lambda} + h \frac{\partial \overline{N}}{\partial \lambda} \right) =$$
$$= \left(\frac{\partial \overline{s}}{\partial \phi} \times \frac{\partial \overline{s}}{\partial \lambda} \right) + h \left(\frac{\partial \overline{s}}{\partial \phi} \times \frac{\partial \overline{N}}{\partial \lambda} \right) + \frac{dh}{d\phi} \left(\overline{N} \times \frac{\partial \overline{s}}{\partial \lambda} \right) + h \frac{dh}{d\phi} \left(\overline{N} \times \frac{\partial \overline{N}}{\partial \lambda} \right) +$$
$$+ h \left(\frac{\partial \overline{N}}{\partial \phi} \times \frac{\partial \overline{s}}{\partial \lambda} \right) + h^2 \left(\frac{\partial \overline{N}}{\partial \phi} \times \frac{\partial \overline{N}}{\partial \lambda} \right) \tag{17}$$

The Weingarten equations give the coordinates of the partial derivatives of the unit normal vector at an arbitrary point *B* of a smooth surface *S* on the tangent plane at the same point. For an arbitrary parameterization (u, v) if $\overline{s}_a$ is the vector equation of a smooth surface *S* and $\overline{N}_a$ the unit normal vector of the surface we have that

$$\frac{\partial \overline{N}_a}{\partial u} = \left(\frac{FM - GL}{EG - F^2} \right) \frac{\partial \overline{s}_a}{\partial u} + \left(\frac{FL - EM}{EG - F^2} \right) \frac{\partial \overline{s}_a}{\partial v} \tag{18a}$$

$$\frac{\partial \overline{N}_a}{\partial v} = \left(\frac{FN - GM}{EG - F^2} \right) \frac{\partial \overline{s}_a}{\partial u} + \left(\frac{FM - EN}{EG - F^2} \right) \frac{\partial \overline{s}_a}{\partial v} \tag{18b}$$

(all the quantities of the above equations are referred to point B. For the above example the arbitrary parameter u is **not** the ellipsoidal coordinate u which is defined in the first paragraph). In our case the parametric lines of the ellipsoid of revolution are also lines of curvature (F = M = 0). Therefore the Weingarten equations become

$$\frac{\partial \overline{N}}{\partial \phi} = -\frac{L}{E} \frac{\partial \overline{s}}{\partial \phi} = -k_1 \frac{\partial \overline{s}}{\partial \phi} \tag{19}$$

$$\frac{\partial \overline{N}}{\partial \lambda} = -\frac{N}{G} \frac{\partial \overline{s}}{\partial \lambda} = -k_2 \frac{\partial \overline{s}}{\partial \lambda} \tag{20}$$

But

$$\left|\frac{\partial \bar{s}}{\partial \phi} \times \frac{\partial \bar{s}}{\partial \lambda}\right| = \sqrt{EG} \tag{21}$$

Using equations (19), (20) and (21), we have the following based on equation (18)

$$\frac{\partial \bar{s}_e}{\partial \phi} \times \frac{\partial \bar{s}_e}{\partial \lambda} = \sqrt{EG}(1 - 2hJ + h^2 K_G)\bar{N} - \sqrt{\frac{G}{E}}\frac{dh}{d\phi}(1 - k_2 h)\frac{\partial \bar{s}}{\partial \phi} \tag{22}$$

where the symbol K_G stands for the Gauss curvature. Using equation (8) we end up with the following differential equation

$$\sqrt{G}\frac{dh}{d\phi}(1 - k_2 h) + \sqrt{EG(1 - k_1 h)^2(1 - k_2 h)^2 + G(1 - k_2 h)^2\left(\frac{dh}{d\phi}\right)^2}\sin\varepsilon = 0 \tag{23}$$

Equation (23) is the differential equation of the generator curve of a normal equipotential surface. Using the approximation

$$\sin\varepsilon = \varepsilon \tag{24}$$

and substituting the necessary quantities in equation (23) we have that

$$\frac{dh}{d\phi} + h\sqrt{E(1 - k_1 h)^2 + \left(\frac{dh}{d\phi}\right)^2}\frac{f^*}{R}\sin 2\phi = 0 \tag{25}$$

This is the final form of the differential equation of the generator curve of the normal equipotential surface passing through point *P*. Given the geodetic coordinates of point *P* as initial condition there exists a solution $h = h(\varphi)$ for the vector equation of the normal equipotential surface. Unfortunately this differential equation cannot be solved analytically. From the above equation it is possible to express the derivative of the function *h* as a function of geodetic latitude and geometric height. The second term of the above equation is positive therefore the derivative of the geometric height must be negative. Developing the second term of the above equation in Newton series (keeping only the linear term) and making the necessary manipulations we have that

$$\left(\frac{dh}{d\phi}\right)^3 - 2\sqrt{E}(1 - k_1 h)^2\frac{dh}{d\phi} - 2E(1 - k_1 h)^3\frac{f^*}{R}\sin 2\phi = 0 \tag{26}$$

Equation (26) is an algebraic equation of third degree (for constant geodetic latitude). The discriminant of the above equation is

$$D = 4E^2(1 - k_1 h)^6\sin^2\varepsilon - \frac{4}{3}E\sqrt{E}(1 - k_1 h)^8 < 0 \tag{27}$$

which is negative therefore it has three real roots. For this case it is convenient to express the solutions in a trigonometric form.

$$\omega_\rho = \arccos\left(\frac{3\sqrt{3}}{2\sqrt{2}}\frac{\sqrt[4]{E}}{(1 - k_1 h)}\sin\varepsilon\right) \quad , \quad r = \sqrt{\frac{8E\sqrt{E}}{27}}(1 - k_1 h)^3 \tag{28}$$

and the three distinct roots are

$$\frac{dh}{d\phi}_{1,2,3} = 2\cdot\sqrt[6]{\frac{8E\sqrt{E}}{27}}(1 - k_1 h)\cos\left\{\frac{1}{3}\arccos\left[\frac{3\sqrt{3}}{2\sqrt{2}}\frac{\sqrt[4]{E}}{(1 - k_1 h)}\frac{f^*}{R}h\sin 2\phi\right] + \frac{2k\pi}{3}\right\},\ k = 0,1,2 \tag{29}$$

The derivative of the geodetic height is negative and not too big, therefore we choose $k = 2$. An alternative form of the solution can be obtained if we set

$$\frac{\pi}{2}-\omega_{\rho}=\arcsin\left(\frac{3\sqrt{3}}{2\sqrt{2}}\frac{\sqrt[4]{E}}{(1-k_1h)}\sin\varepsilon\right) \tag{30}$$

and

$$\begin{aligned}\frac{dh}{d\phi}&=2\cdot\sqrt[6]{\frac{8E\sqrt{E}}{27}}(1-k_1h)\cos\left\{\frac{4\pi}{3}+\frac{\pi}{6}-\frac{1}{3}\arcsin\left[\frac{3\sqrt{3}}{2\sqrt{2}}\frac{\sqrt[4]{E}}{(1-k_1h)}\frac{f^*}{R}h\sin 2\phi\right]+\frac{4\pi}{3}\right\}=\\&=2\cdot\sqrt[6]{\frac{8E\sqrt{E}}{27}}(1-k_1h)\cos\left\{\frac{3\pi}{2}-\frac{1}{3}\arcsin\left[\frac{3\sqrt{3}}{2\sqrt{2}}\frac{\sqrt[4]{E}}{(1-k_1h)}\frac{f^*}{R}h\sin 2\phi\right]+\frac{4\pi}{3}\right\}=\\&=-2\cdot\frac{\sqrt{2}}{\sqrt{3}}\sqrt[4]{E}(1-k_1h)\sin\left\{\frac{1}{3}\arcsin\left[\frac{3\sqrt{3}}{2\sqrt{2}}\frac{\sqrt[4]{E}}{(1-k_1h)}\frac{f^*}{R}h\sin 2\phi\right]\right\}\end{aligned} \tag{30a}$$

After some manipulations (using also power series for the functions “sin” and “arcsin”) we have eventually that

$$\frac{dh}{d\phi}=-\sqrt{E}\frac{f^*}{R}h\sin 2\phi\left[1+\frac{\sqrt{E}}{(1-k_1h)^2}\frac{(f^*)^2}{R^2}h^2\sin^2 2\phi\right] \tag{30b}$$

By derivating the above formula and keeping the significant terms, the second order derivative of geometric height is estimated from the following relation

$$\frac{d^2h}{d\phi^2}=-E\sin^2 2\phi\frac{(f^*)^2}{R^2}h-2\cos 2\phi\sqrt{E}\frac{f^*}{R}h-E\frac{\sin^3 2\phi}{(1-k_1h)^2}\frac{(f^*)^3}{R^3}h^3 \tag{31}$$

Step c): Formulation of the fundamental quantities of the normal equipotential surface passing through point P in geodetic coordinates

After some manipulations with the help of relations (5), (6) and (30b) we get the following formula

$$E_e=\langle\frac{\partial\bar{s}_e}{\partial\phi},\frac{\partial\bar{s}_e}{\partial\phi}\rangle=E(1-hk_1)^2+\left(\frac{dh}{d\phi}\right)^2 \tag{32}$$

or

$$E_e=E(1-k_1h)^2+\left\{\sqrt{E}\frac{f^*}{R}h\sin 2\phi\left[1+\frac{\sqrt{E}}{(1-k_1h)^2}\frac{(f^*)^2}{R^2}h^2\sin^2 2\phi\right]\right\}^2 \tag{32a}$$

In addition

$$F_e=\langle\frac{\partial\bar{s}_e}{\partial\phi},\frac{\partial\bar{s}_e}{\partial\lambda}\rangle=0 \tag{33}$$

$$G_e=\langle\frac{\partial\bar{s}_e}{\partial\lambda},\frac{\partial\bar{s}_e}{\partial\lambda}\rangle=G(1-hk_2)^2 \tag{34}$$

Derivating the relations (5) and (6) we have that

$$\frac{\partial^2\bar{s}_e}{\partial\phi^2}=\frac{\partial^2\bar{s}}{\partial\phi^2}+\frac{d^2h}{d\phi^2}\bar{N}+2\frac{dh}{d\phi}\frac{\partial\bar{N}}{\partial\phi}+h\frac{\partial^2\bar{N}}{\partial\phi^2} \tag{35}$$

$$\frac{\partial^2 \overline{s}_e}{\partial \lambda^2} = \frac{\partial^2 \overline{s}}{\partial \lambda^2} + h\frac{\partial^2 \overline{N}}{\partial \lambda^2} \tag{36}$$

The determination of the fundamental quantities L_e, M_e, N_e involves the second order partial derivatives of the unit normal vector of the ellipsoid. Derivation of the Weingarten equations yields

$$\frac{\partial^2 \overline{N}}{\partial \phi^2} = \left(-\frac{1}{E}\frac{\partial L}{\partial \phi} + \frac{1}{E}k_1\frac{\partial E}{\partial \phi}\right)\frac{\partial \overline{s}}{\partial \phi} - k_1\frac{\partial^2 \overline{s}}{\partial \phi^2} \tag{37a}$$

$$\frac{\partial^2 \overline{N}}{\partial \lambda^2} = -k_2\frac{\partial^2 \overline{s}}{\partial \lambda^2} \tag{37b}$$

The Christoffel symbols of the second kind $\Gamma_{1,11}$ and $\Gamma_{1,22}$ [7] in our case become

$$\Gamma_{1,11} = \langle \frac{\partial \overline{s}}{\partial \phi}, \frac{\partial^2 \overline{s}}{\partial \phi^2} \rangle = \frac{1}{2}\frac{\partial E}{\partial \phi} \tag{38a}$$

$$\Gamma_{1,22} \langle \frac{\partial \overline{s}}{\partial \phi}, \frac{\partial^2 \overline{s}}{\partial \lambda^2} \rangle = -\frac{1}{2}\frac{\partial G}{\partial \phi} \tag{38b}$$

Relations (38a) and (38b) are needed for the formulations of the fundamental quantities L_e, and N_e.

$$L_e = \langle \overline{N}_e, \frac{\partial^2 \overline{s}_e}{\partial \phi^2} \rangle = \langle \overline{N}\cos\varepsilon + \frac{1}{\left|\frac{\partial \overline{s}}{\partial \phi}\right|}\frac{\partial \overline{s}}{\partial \phi}\sin\varepsilon \ , \ \frac{\partial^2 \overline{s}}{\partial \phi^2} + \frac{d^2 h}{d\phi^2}\overline{N} + 2\frac{dh}{d\phi}\frac{\partial \overline{N}}{\partial \phi} + h\frac{\partial^2 \overline{N}}{\partial \phi^2} \rangle \tag{39}$$

Therefore

$$L_e = \left[(1-hk_1)L + \frac{d^2 h}{d\phi^2}\right]\cos\varepsilon + \left(\frac{1}{2\sqrt{E}}\frac{\partial E}{\partial \phi} - 2k_1\frac{dh}{d\phi}\sqrt{E} - h\frac{\partial k_1}{\partial \phi}\sqrt{E} - \frac{1}{2\sqrt{E}}\frac{\partial E}{\partial \phi}k_1 h\right)\sin\varepsilon \tag{40}$$

$$M_e = 0 \tag{41}$$

Accordingly

$$N_e = (1-hk_2)N\cos\varepsilon - \frac{1}{2\sqrt{E}}(1-hk_2)\frac{\partial G}{\partial \phi}\sin\varepsilon \tag{42}$$

We remind that we confine ourselves on the Earth's physical surface therefore we set

$$\cos\varepsilon = 1 - \frac{\varepsilon^2}{2} \tag{43}$$

$$\sin\varepsilon = \varepsilon \tag{44}$$

Equation (40) becomes

$$L_e = (1-k_1h)L + \frac{d^2h}{d\phi^2} + \frac{1}{2}\frac{f^*}{R\sqrt{E}}\sin 2\phi\frac{\partial E}{\partial\phi}h - \left\{\frac{1}{2}\frac{(f^*)^2}{R^2}L\sin^2 2\phi - \frac{1}{2\sqrt{E}}\frac{f^*}{R}k_1\frac{\partial E}{\partial\phi}\sin 2\phi + \right.$$
$$\left. + 2k_1E\frac{(f^*)^2}{R^2}\sin^2 2\phi\left[1+\frac{\sqrt{E}}{(1-k_1h)^2}\frac{(f^*)^2}{R^2}h^2\sin^2 2\phi\right] - \sqrt{E}\frac{f^*}{R}\frac{\partial k_1}{\partial\phi}\sin 2\phi\right\}h^2 - \left[Lk_1\frac{(f^*)^2}{R^2}\sin^2 2\phi\right]h^3 \tag{44}$$

or neglecting some small terms

$$L_e = (1-k_1h)L + \left[\frac{1}{2}\frac{f^*}{R\sqrt{E}}\sin 2\phi\frac{\partial E}{\partial\phi} - E\sin^2 2\phi\frac{(f^*)^2}{R^2} - 2\cos 2\phi\sqrt{E}\frac{f^*}{R}\right]h -$$
$$-\left[\frac{1}{2}\frac{(f^*)^2}{R^2}L\sin^2 2\phi - \sqrt{E}\frac{f^*}{R}\frac{\partial k_1}{\partial\phi}\sin 2\phi\right]h^2 - E\frac{(f^*)^2\sin^2 2\phi}{R^2}\left(1-\frac{f^*}{R}\sin 2\phi\right)h^3 \tag{46}$$

Equation (42) becomes

$$N_e = (1-k_2h)N - (1-k_2h)N\frac{1}{2}\sin^2 2\phi\frac{(f^*)^2}{R^2}h^2 - (1-k_2h)\frac{1}{2\sqrt{E}}\frac{\partial G}{\partial\phi}\sin 2\phi\frac{f^*}{R}h \tag{47a}$$

or

$$N_e = (1-k_2h)N - \left[\frac{1}{2\sqrt{E}}\frac{\partial G}{\partial\phi}\frac{f^*}{R}\sin 2\phi\right]h - \left[N\frac{1}{2}\frac{(f^*)^2}{R^2}\sin^2 2\phi\right]h^2 + N\frac{k_2\sin^2 2\phi}{2}\frac{(f^*)^2}{R^2}h^3 \tag{47b}$$

TABLE 1 FUNDAMENTAL QUANTITIES OF THE EQUIPOTENTIAL SURFACE

E_e	$E(1-k_1h)^2 + \left\{\sqrt{E}\frac{f^*}{R}h\sin 2\phi\left[1+\frac{\sqrt{E}}{(1-k_1h)^2}\frac{(f^*)^2}{R^2}h^2\sin^2 2\phi\right]\right\}^2$
F_e	0
G_e	$G(1-hk_2)^2$
L_e	$(1-k_1h)L + \left[\frac{1}{2}\frac{f^*}{R\sqrt{E}}\sin 2\phi\frac{\partial E}{\partial\phi} - E\sin^2 2\phi\frac{(f^*)^2}{R^2} - 2\cos 2\phi\sqrt{E}\frac{f^*}{R}\right]h -$ $-\left[\frac{1}{2}\frac{(f^*)^2}{R^2}L\sin^2 2\phi - \sqrt{E}\frac{f^*}{R}\frac{\partial k_1}{\partial\phi}\sin 2\phi\right]h^2 - E\frac{(f^*)^2\sin^2 2\phi}{R^2}\left(1-\frac{f^*}{R}\sin 2\phi\right)h^3$
M_e	0
N_e	$(1-k_2h)N - \left[\frac{1}{2\sqrt{E}}\frac{\partial G}{\partial\phi}\frac{f^*}{R}\sin 2\phi\right]h - \left[N\frac{1}{2}\frac{(f^*)^2}{R^2}\sin^2 2\phi\right]h^2 + N\frac{k_2\sin^2 2\phi}{2}\frac{(f^*)^2}{R^2}h^3$

Table 1 shows the final form of the fundamental quantities of the normal equipotential surface passing through point *P*. We remind that the letters *E*, *F*, *G*, *L*, *M*, *N*, are the fundamental quantities of the ellipsoid.

Step d): Formulation of the mean curvature of the normal equipotential surface passing through point *P*.

Now we are ready to express the mean curvature of the normal equipotential surface passing through point *P* as a function of geodetic latitude and geometric height. We have that

$$J_e = \frac{1}{2}\left(\frac{L_e}{E_e} + \frac{N_e}{G_e}\right) \tag{48}$$

or

$$J_e = \frac{1}{2}\left[\frac{(1-k_1 h)L}{E(1-k_1 h)^2 + \left(\frac{dh}{d\phi}\right)^2} + \frac{k_2}{(1-k_2 h)} + \frac{C_1(\phi,h)}{E(1-k_1 h)^2 + \left(\frac{dh}{d\phi}\right)^2} + \frac{C_2(\phi,h)}{G(1-k_2 h)^2}\right] \tag{48a}$$

where

$$\begin{aligned} C_1(\phi,h) = {} & \frac{1}{2}\frac{f^*}{R\sqrt{E}}\sin 2\phi \frac{\partial E}{\partial \phi} h - E\sin^2 2\phi \frac{(f^*)^2}{R^2} h - 2\cos 2\phi \sqrt{E}\frac{f^*}{R} h - \\ & -\left[\frac{1}{2}\frac{(f^*)^2}{R^2} L\sin^2 2\phi - \sqrt{E}\frac{f^*}{R}\frac{\partial k_1}{\partial \phi}\sin 2\phi\right] h^2 - E\frac{(f^*)^2 \sin^2 2\phi}{R^2}\left(1 - \frac{f^*}{R}\sin 2\phi\right) h^3 \end{aligned} \tag{49}$$

$$C_2(\phi,h) = -\left[\frac{1}{2\sqrt{E}}\frac{\partial G}{\partial \phi}\frac{f^*}{R}\sin 2\phi\right] h - \left[N\frac{1}{2}\frac{(f^*)^2}{R^2}\sin^2 2\phi\right] h^2 + N\frac{k_2 \sin^2 2\phi}{2}\frac{(f^*)^2}{R^2} h^3 \tag{50}$$

For $h = 0$, equation (44) describes the mean curvature of the ellipsoid of revolution. The Bruns equation for an arbitrary point on the ellipsoid is written as

$$\frac{\partial \gamma}{\partial h} = -2\omega^2 - 2\gamma J_e(\phi,\lambda,0) \tag{51}$$

For an arbitrary point P above the ellipsoid we obtain

$$\left.\frac{\partial \gamma}{\partial z}\right|_P = -2\omega^2 - 2\gamma(P) J_e(P) \tag{52}$$

where the letter "z" represents the vertical axis to the normal equipotential surface at point P. The determination of normal gravity at point P for low geometric heights above the ellipsoid can be done by using a series expansion

$$\gamma(P) = \gamma(0) + \left.\frac{\partial \gamma}{\partial h}\right|_{h=0} h + \frac{1}{2}\left.\frac{\partial^2 \gamma}{\partial h^2}\right|_{h=0} h^2 \tag{53}$$

where [12], (all quantities on the right side are referred to geometric height $h = 0$)

$$\left.\frac{\partial^2 \gamma}{\partial h^2}\right|_{h=0} = -\gamma({k_1}^2 + {k_2}^2) + \frac{\partial^2 \gamma}{\partial y^2} \tag{54}$$

with

$$\gamma \equiv \gamma(\phi) = \frac{a\gamma_\alpha \cos^2\phi + b\gamma_b \sin^2\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}} \tag{55}$$

For the determination of γ_α, and γ_b see [3]. The letter "y" in equation (54) stands for the axis which passes through point Q on the ellipsoid of revolution, it points towards the north pole of the ellipsoid and it is tangent to the meridian at point Q. We also have that [12]

$$\frac{\partial \phi}{\partial y} = \frac{1}{\frac{\partial R_2}{\partial \phi}\left(\frac{b^2}{2a^2} - \frac{1}{2}\right)\sin 2\phi + R_2\left(\frac{b^2}{a^2}\cos^2\phi + \sin^2\phi\right)} \tag{56}$$

For large values of geometric height we can use normal gravity formulae either from Hirvonen [9] or from Grafarend and Ardalan [13].

III. CONCLUSIONS

A method of determining the vertical derivative of normal gravity for points on the Earth's physical surface was presented. To do this we chose to express Bruns formula in geodetic coordinates using the fundamental quantities of a normal equipotential surface passing through a point P. These quantities were expressed as functions of geodetic coordinates. The angle ε between the vertical line to the ellipsoid passing through point P and the vertical line to the normal equipotential surface at the same point was considered as a parameter in the equation of the normal equipotential surface. In the sequel we formulated the function J_e which describes the mean curvature of the normal equipotential surface in geodetic coordinates. The advantage of this function is that it makes possible the determination of the mean curvature of a normal equipotential surface at a point with known geodetic coordinates without knowing the value of normal potential at this point. Having the function J_e we were able to form Bruns equation in geodetic coordinates and determine the vertical derivative of normal gravity on and above the ellipsoid of revolution (confining ourselves on the Earth's physical surface). Finally it is worth mentioning that the presented formula for the vertical gradient of normal gravity may be used for a better determination of the normal part of the vertical gradient of the actual gravity.

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