Initial Study of Normal Isocurvature Surfaces and Their Relation to Partial Derivatives of Plumb Line Curvature

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***Abstract*-This work aimed to study isocurvature surfaces of Earth's normal gravity field and their relation to partial derivatives of a plumb line curvature. An isocurvature surface of a gravity field is a surface along which the value of the plumb line curvature is constant. The normal gravity field is a symmetrical gravity field; therefore, isocurvature surfaces are surfaces of revolution. To study an isocurvature surface, special assumptions are made to form a vector equation, which will hold only for a small coordinate patch of the isocurvature surface. The gradient of a normal plumb line curvature is vertical to the isocurvature surface pointing to the direction along which the curvature of the plumb line decreases or increases the most. In order to show the significance of isocurvature surfaces, it was shown that it is possible to determine the value of the surface derivative of a plumb line's curvature without differentiating the original complicated function of a plumb line curvature.**

Keywords- Plumb Lines; Curvature; Normal Gravity Field; Plumbline Curvature; Isocurvature Surface

I. INTRODUCTION

In Physical Geodesy and when studying vector fields in general, scientists define geometrical objects related to invariant quantities of the vector field of their respective interest. To illustrate these concepts as they apply to geodesy, [1] and [2], for instance, presented three characteristic examples: isozenithal lines, equipotential surfaces, and equigravitational surfaces. Specifically, along isozenithal lines, the angle between the gravity vector and the equatorial plane is constant, while on an equipotential surface, it is the gravity potential that is constant. On an isogravitational surface, the modulus of gravity vector is constant. All of these quantities, which remain constant on these geometrical objects, are also invariant quantities. Another important invariant quantity related to the gravity field is the plumb line curvature. Isocurvature lines were defined by [3]. As a theoretical application, two different families of normal isocurvature lines were studied: on a meridian plane and on a parallel plane. In this work, normal isocurvature surfaces were studied. Earth's normal gravity field, with its rotational symmetry, simplified the desirable equations for this study. The first step in this study involved the determination of a suitable coordinate system for the local parameterization of the isocurvature surface around a chosen point P. This was required to show that the isocurvature surface's parametric lines were also lines of curvature. The second step introduced a suitable function describing the behavior of the plumb line curvature function in the coordinate patch of the isocurvature surface. In the third step, the vector equation of the isocurvature surface was formulated and its fundamental quantities, which are related to the Gauss curvature, were determined. Finally, we describe an algebraic method for the determination of the surface derivative (along the North – South direction) of the plumb line curvature was determined without differentiating the complicated function of plumb line curvature.

II. METHODOLOGY

Consider a Cartesian equatorial system (X, Y, Z) whose origin is situated at Earth's center of mass. The Z-axis is Earth's mean axis of rotation, the X-axis is the intersection of the meridian plane of Greenwich, and the equator's plane and the Y-axis make the system right-handed. If k(X, Y, Z) is the function for the plumb line curvature, then an isocurvature surface of curvature value k0 can be described by the following equation:

$$k(X,Y,Z)=k_0 \tag{1}$$

Due to the rotational symmetry of the normal gravity field, the isocurvature surfaces are *surfaces of revolution*. The plumb line curvature is a complicated function, which contains first- and second- order partial derivatives of the normal potential. This makes it difficult to use it for the study of an isocurvature surface of a given value of plumb line curvature *k*. To overcome this difficulty and to form a simple vector equation for the isocurvature surface, in previous work, [3] defined the function k_{1a},

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$$k_{1a}(y_1,h_1)=\frac{\left|c_1y_1+c_2h_1+c_3+c_7{y_1}^2+c_8y_1h_1+c_9{h_1}^2\right|}{c_4[1+2(c_5y_1+c_6h_1+c_{10}{y_1}^2+c_{11}y_1h_1+c_{12}{h_1}^2)]^{3/2}} \tag{2}$$

and provided full expressions for the coefficients c_i (i = 1,2,…,12). Eq. (2) involves two local Cartesian systems (x, y, h) and (x_1, y_1, h_1). Let P be a point above the ellipsoid and Q is its projection to the ellipsoid along the vertical line to the ellipsoid passing through point P. The first is centered at point Q, the x-axis is tangent to the local parallel positive to the east, the y-axis is tangent to the local meridian positive to the north, and the h-axis is the vertical line to the ellipsoid passing through the point of interest P. The second system (x_1, y_1, h_1) is a parallel transport of the previous one and its center is at point P.

Let D be the interior of a circle on the meridian plane of P, which has its center at point P and radius $\varepsilon = 1\ m$. The equation for this circle is:

$$ {y_1}^2+{h_1}^2=\varepsilon^2 \tag{3}$$

With Eq. (2), it is now possible to determine the curvature of a plumb line passing through an arbitrary point Q in set D at a specific point. The intersection between the meridian plane of P and the isocurvature surface passing through the same point is an isocurvature line. This isocurvature line in set D has an equation of the form:

$$k_{1a}(y_1,h_1)=k_{1a}(P) \tag{4}$$

From the initial value theorem there exists a function

$$a_{is}(y_1)\equiv h_1(y_1)\ \ or\ \ a_{is}(y)\equiv h(y) \tag{5}$$

that describes the isocurvature line in set D. This allows a vector equation to be formed for a coordinate patch of the isocurvature surface passing through P. If (φ, λ) represent the geodetic latitude and the geodetic longitude, then a parameterization of the isocurvature surface is:

$$\bar{s}(y,\lambda)=(X(y,\lambda),Y(y,\lambda),Z(y,\lambda)) \tag{6}$$

and

$$X(y,\lambda)=(pr_1\circ\bar{s})(y,\lambda)=((N+h(y))\cos\phi_P\cos\lambda-\sin\phi_P\cdot(y\cos\lambda)+\cos\phi_P\cdot(\cos\lambda\cdot h(y))$$

$$Y(y,\lambda)=(pr_2\circ\bar{s})(y,\lambda)=((N+h(y))\cos\phi_P\sin\lambda-\sin\phi_P\cdot(y\sin\lambda)+\cos\phi_P\cdot(\sin\lambda\cdot h(y)) \tag{7a}$$

$$Z(y,\lambda)=(pr_3\circ\bar{s})(y,\lambda)=(\frac{b^2}{a^2}(N+h(y))\sin\phi_P+\cos\phi_P\cdot y+\sin\phi_P\cdot h(y))$$

where [4]

$$N\equiv N(\phi_P)=k_2(P)=\frac{b}{a^2}(1+e'^2\cos^2\phi_P)^{\frac{1}{2}} \tag{7b}$$

is the normal radius of the curvature of the ellipsoid taken in the direction of the prime vertical and

$$e'=\frac{E}{b}=\frac{\sqrt{a^2-b^2}}{b} \tag{7c}$$

In the remainder of the paper, in order to be more concise, the letter "k" will be used instead of the symbol "k_{1a}" when the partial derivatives of the plumb line curvature are used.

III. STUDY OF THE ISOCURVATURE SURFACES

The intersection of a family of isocurvature surfaces and a meridian plane are plane curves that are isocurvature lines. The normal plumb lines on the equatorial plane are straight lines, and the plumb line that is vertical to the North Pole is also a straight line. Therefore, as the value of the plumb line curvature tends to zero, the isocurvature surfaces tend asymptotically to the equatorial plane and the Z – axis, as depicted in Fig. 1.

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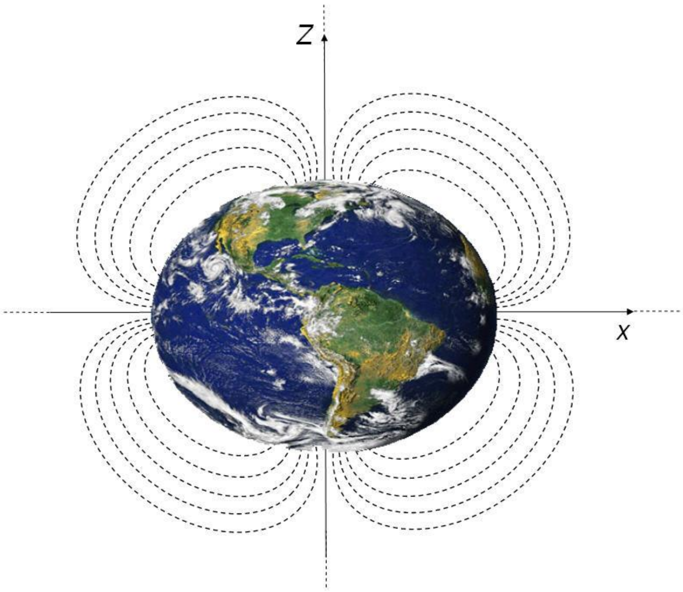

Fig. 1 The isocurvature lines of the normal gravity field on the meridian plane

The parameterization of an isocurvature surface has two variables, one linear *(y)* and one angular *(λ)*. The quantities of the first fundamental forms are given by the following expressions:

$$E = \langle \frac{\partial \overline{s}}{\partial y}, \frac{\partial \overline{s}}{\partial y} \rangle = 1 + \frac{k_y^2}{k_h^2} \tag{8}$$

$$F = \langle \frac{\partial \overline{s}}{\partial y}, \frac{\partial \overline{s}}{\partial \lambda} \rangle = 0 \tag{9}$$

$$G = \langle \frac{\partial \overline{s}}{\partial \lambda}, \frac{\partial \overline{s}}{\partial \lambda} \rangle = (N + h(y))^2 \cos^2 \phi_P + y^2 \sin^2 \phi_P - (N + h(y)) y \sin 2\phi_P \tag{10}$$

The normal vector to the surface is:

$$\begin{aligned} \overline{N}(y,\lambda) &= (N_1(y,\lambda), N_2(y,\lambda), N_3(y,\lambda)) = \\ &= (-N\cos^2\phi_P\cos\lambda + N\cos\phi_P\sin\phi_P\cos\lambda\frac{k_y}{k_h} + y\cos\phi_P\sin\phi_P\cos\lambda - \\ &\quad - y\sin^2\phi_P\cos\lambda\frac{k_y}{k_h} - h(y)\cos^2\phi_P\cos\lambda + h(y)\frac{k_y}{k_h}\cos\phi_P\sin\phi_P\cos\lambda, \\ &\quad - N\cos^2\phi_P\sin\lambda + N\cos\phi_P\sin\phi_P\sin\lambda\frac{k_y}{k_h} + y\cos\phi_P\sin\phi_P\sin\lambda - \\ &\quad y\sin^2\phi_P\sin\lambda\frac{k_y}{k_h} - h(y)\cos^2\phi_P\sin\lambda + h(y)\frac{k_y}{k_h}\cos\phi_P\sin\phi_P\sin\lambda, \\ &\quad - N\cos\phi_P\sin\phi_P + y\sin^2\phi_P - h(y)\cos\phi_P\sin\phi_P - N\frac{k_y}{k_h}\cos^2\phi_P + \\ &\quad + y\frac{k_y}{k_h}\cos\phi_P\sin\phi_P - h(y)\frac{k_y}{k_h}\cos^2\phi_P) \end{aligned} \tag{11}$$

For the quantities of the second fundamental form there is:

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$$M = \frac{1}{\left|\overline{N}\right|}\langle \overline{N}, \frac{\partial^2 \overline{s}}{\partial y \partial \lambda}\rangle = 0 \tag{12}$$

Therefore, since $F = M = 0$, the parametric lines $y = c$ and $\lambda = c$ are lines of curvature. The curve $\lambda = c$ passing through point P is an isocurvature line and its equation is $h=h(y)$. Since it is a plane curve its curvature is equal to

$$K = \frac{d^2h}{dy}\frac{1}{\left(1+ {dh}/{dy}\right)^{3/2}} \tag{13}$$

and the principal curvature k_1 along this direction is equal to

$$k_1 = K = \frac{\left|k_h^3\right|(-k_{yy}k_h^2 + 2k_{yh}k_y k_h - k_{hh}k_y^2)}{(k_y^2 + k_h^2)^{3/2}} \tag{14}$$

In order to find the second principal curvature k_2 (along the direction $y = c$), the Meusnier's theorem is used, leading to the following expression for k_2:

$$k_2 = \frac{1}{(X^2 + Y^2)^{1/2}}\frac{(N_1^2 + N_2^2)^{1/2}}{\left|\overline{N}\right|} \tag{15}$$

The first term in (15) describes the curvature of the parallel circle and the second term is the cosine of the angle between the normal vector and the equatorial plane. Furthermore, because the parametric lines of the isocurvature surfaces are lines of curvature, then:

$$k_1 = \frac{L}{E} \tag{16}$$

$$k_2 = \frac{N}{G} \tag{17}$$

Using Relations (14) and (15), the other two quantities of the second fundamental form can be determined, i.e.

$$L = k_1 \cdot E \tag{18}$$

$$N = k_2 \cdot G \tag{19}$$

The Gauss curvature of the isocurvature surface can now be expressed as

$$K_G = \frac{\left|k_h^3\right|(-k_{yy}k_h^2 + 2k_{yh}k_y k_h - k_{hh}k_y^2)(N_1^2 + N_2^2)^{1/2}}{(k_y^2 + k_h^2)^{3/2}(X^2 + Y^2)^{1/2}\left|\overline{N}\right|} \tag{20}$$

From the study of the isocurvature surfaces it can be seen that it is not possible to write all the above relations in an explicit form. That is because all these expressions involve the unknown term $h(y)$. However, it is still possible, if it is required, to carry out the necessary calculations at a specific point of the isocurvature surface by solving equation $k(y, h) = k_P$ numerically for a given y and, hence to find the value of the unknown term $h(y)$ at a specific point of interest.

IV. AN APPLICATION RELATED TO ISOCURVATURE SURFACES

The problem at hand is the following:

Given the geodetic coordinates (φ_P, λ_P, h_P) of a point P on Earth's physical surface, find the partial derivatives k_y and k_h of the plumb line curvature at the same point.

The typical procedure to find these partial derivatives [5] is to derivate the plumb lines' curvature function

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$$\begin{aligned}k(P) = \Big\{&\big[U_y(U_{xh}U_x + U_{yh}U_y) + U_h(U_{hh}U_y - U_{xy}U_x - U_{yy}U_y - U_{yh}U_h)\big]_P^{\,2} + \\ &+\big[U_h({U_{xx}}^2U_x + U_{xy}U_y + U_{xh}U_h) - U_x(U_{xh}U_x + U_{yh}U_y + U_{hh}U_h)\big]_P^{\,2} + \\ &+\big[U_x(U_{xy}U_x + U_{yy}U_y + U_{yh}U_h + U_{xx}U_x) - U_y(U_{xy}U_y - U_{xh}U_h)\big]_P^{\,2}\Big\}^{\frac{1}{2}} \cdot \\ &\cdot \frac{1}{({U_x}^2 + {U_y}^2 + {U_h}^2)_P^{\frac{3}{2}}}\end{aligned} \tag{21}$$

These partial derivatives are going to be found in a different way. Let Q be the projection of point P on the ellipsoid along the vertical line to the ellipsoid passing through point P. Since it is restricted close to the ellipsoid, the following formula [6] for the determination of the plumb line curvature at point P is used:

$$k(P) = \frac{\gamma}{[\gamma - 2(\omega^2 + \gamma J)h_P]} \cdot \left\{k + \left[k(k_1 - k_2) + 2\frac{\partial J}{\partial y}\right]h_P\right\} \tag{21a}$$

where the quantities of the above formula (principal curvatures k_1, k_2, and mean curvature J) are determined at point Q. The local Cartesian system (x, y, h) for the parameterization of the isocurvature surface passing through point P is centered at point Q. The x-axis is tangent to the local parallel positive to the east, the y-axis is tangent to the local meridian positive to the north, and the h-axis is the vertical line to the ellipsoid passing through the point of interest P. Using Eq. (21), is possible to find the partial derivative k_h at point P, which is equal to:

$$\begin{aligned}k_h(P) \equiv k_h = &\frac{2(\omega^2 + \gamma J)}{[\gamma - 2(\omega^2 + \gamma J)h_P]^2}\left\{k + \left[k(k_1 - k_2) + 2\frac{\partial J}{\partial y}\right]h_P\right\} + \\ &+ \frac{\gamma}{[\gamma - 2(\omega^2 + \gamma J)h_P]}\left[k(k_1 - k_2) + 2\frac{\partial J}{\partial y}\right]\end{aligned} \tag{22}$$

and

$$\frac{\partial \phi}{\partial y} = \frac{1}{\dfrac{\partial}{\partial \phi}\left(\dfrac{1}{k_1}\right)\left(\dfrac{b^2}{2a^2} - \dfrac{1}{2}\right)\sin 2\phi_Q + \dfrac{1}{k_1}\left(\dfrac{b^2}{a^2}\cos^2\phi_Q + \sin^2\phi_Q\right)} \tag{23}$$

$$k_1 = \frac{b}{a^2}(1 + e'^2\cos^2\phi_Q)^{\frac{3}{2}} \tag{24}$$

$$k_2 = \frac{b}{a^2}(1 + e'^2\cos^2\phi_Q)^{\frac{1}{2}} \tag{25}$$

$$J = \frac{1}{2}(k_1 + k_2) \tag{26}$$

$$k = \frac{1}{\gamma}\frac{\partial \gamma}{\partial y}\bigg|_Q \tag{27}$$

$$\gamma = \frac{a\gamma_\alpha \cos^2\phi_Q + b\gamma_b \sin^2\phi_Q}{\sqrt{a\cos^2\phi_Q + b\sin^2\phi_Q}} \tag{28}$$

For the determination of normal gravity at equator and at pole (γ_a and γ_b, respectively) see [4]. The remaining partial derivative k_y (derivative along North – South direction) at point P will be found with the help of the theory of isocurvature surfaces. The normal vector to the isocurvature surface at point P is equal to

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$$\begin{aligned}\overline{N}(P) = (&-N\cos^2\phi_P\cos\lambda + N\cos\phi_P\sin\phi_P\cos\lambda\frac{k_y}{k_h} - h(0)\cos^2\phi_P\cos\lambda + \\ &+h(0)\frac{k_y}{k_h}\cos\phi_P\sin\phi_P\cos\lambda, \\ &-N\cos^2\phi_P\sin\lambda + N\cos\phi_P\sin\phi_P\sin\lambda\frac{k_y}{k_h} - h(0)\cos^2\phi_P\sin\lambda + \\ &+h(0)\frac{k_y}{k_h}\cos\phi_P\sin\phi_P\sin\lambda, \\ &-N\cos\phi_P\sin\phi_P - h(0)\cos\phi_P\sin\phi_P - N\frac{k_y}{k_h}\cos^2\phi_P - h(0)\frac{k_y}{k_h}\cos^2\phi_P)\end{aligned} \tag{29}$$

where $h(0) = h_P$. The angle Φ between the normal vector at point P and the meridian plane is determined by the following formula:

$$\tan\Phi_P = \frac{|N_3|}{|N_1{}^2 + N_2{}^2|} \tag{30}$$

From Relation (29) there is

$$|N_1{}^2 + N_2{}^2| = \sqrt{N_1{}^2 + N_2{}^2} = (N+h)_P\cos\phi_P\left|\left(\cos\phi_P - \frac{k_y}{k_h}\sin\phi_P\right)\right| \tag{31}$$

$$|N_3| = (N+h)_P\cos\phi_P\left|\left(\sin\phi_P + \frac{k_y}{k_h}\cos\phi_P\right)\right| \tag{32}$$

Therefore,

$$\tan\Phi_P = \frac{\left|\sin\phi_P + \frac{k_y}{k_h}\cos\phi_P\right|}{\left|\cos\phi_P - \frac{k_y}{k_h}\sin\phi_P\right|} = |c_P| \tag{33}$$

When $0 < \varphi_P < 90^0$, the partial derivative $k_y > 0$ tends to zero as φ_P tends to 90^0; hence, k_y / k_h is negative. The denominator is always positive. Therefore, the change of sign of the nominator results in a change of sign of c_P. The relation between the two partial derivatives is given by the following formula:

$$k_y = \frac{c_P\cos\phi_P - \sin\phi_P}{\cos\phi_P + c_P\sin\phi_P}k_h \tag{34}$$

For the determination of c_P a point $Q_1 = (\varphi_{Q1}, \lambda_Q, 0)$ is chosen on the ellipsoid that is close to point P. Using Eq. (21) a point $P_1 = (\varphi_{Q1}, \lambda_Q, h_{P1})$ is determined such that $k(P_1) = k(P)$. This can be done starting from the initial value $h' = h_P$. Then a suitable value $\delta h'$ is found such that $h'_{P1} = h_P + \delta h'$ or $h'_{P1} = h_P - \delta h'$. The coordinates of point P_1 in (X, Y, Z) system can be found [7] from the following transformation:

$$\begin{bmatrix} X_{P_1} \\ Y_{P_1} \\ Z_{P_1} \end{bmatrix} = \begin{bmatrix} X_{Q_1} \\ Y_{Q_1} \\ Z_{Q_1} \end{bmatrix} + \begin{bmatrix} -\sin\lambda_P & -\sin\phi_{Q_1}\cos\lambda_Q & \cos\phi_{Q_1}\cos\lambda_Q \\ \cos\lambda_Q & -\sin\phi_{Q_1}\sin\lambda_Q & \cos\phi_{Q_1}\sin\lambda_Q \\ 0 & \cos\phi_{Q_1} & \sin\phi_{Q_1} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ h'_{P_1} \end{bmatrix} \tag{35}$$

The transformation of the Cartesian coordinates of point P_1 expressed in the (X, Y, Z) system to the local system (x, y, h) centered at point Q is given by the following:

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$$\begin{bmatrix} x_{P_1} \\ y_{P_1} \\ h_{P_1} \end{bmatrix} \equiv \begin{bmatrix} x_1 \\ y_1 \\ h_1 \end{bmatrix} = \begin{bmatrix} -\sin\lambda_P & \cos\lambda_Q & 0 \\ -\sin\phi_Q\cos\lambda_Q & -\sin\phi_Q\sin\lambda_Q & \cos\phi_Q \\ \cos\phi_Q\cos\lambda_Q & \cos\phi_Q\sin\lambda_Q & \sin\phi_Q \end{bmatrix} \begin{bmatrix} X_{P_1} - X_Q \\ Y_{P_1} - Y_Q \\ Z_{P_1} - Z_Q \end{bmatrix} \tag{36}$$

Since the difference $\delta\varphi = \varphi_{Q1} - \varphi_Q$ was chosen to be small, the vector (0, y_1, $h_P - h_1$) is assumed to be tangent to the isocurvature surface at point P. A vertical vector to it is (0, $-h_P + h_1$, y_1). Using Transformation (35) there is

$$\begin{bmatrix} X_v \\ Y_v \\ Z_v \end{bmatrix} = \begin{bmatrix} X_Q \\ Y_Q \\ Z_Q \end{bmatrix} + \begin{bmatrix} -\sin\lambda_P & -\sin\phi_Q\cos\lambda_Q & \cos\phi_Q\cos\lambda_Q \\ \cos\lambda_Q & -\sin\phi_Q\sin\lambda_Q & \cos\phi_Q\sin\lambda_Q \\ 0 & \cos\phi_Q & \sin\phi_Q \end{bmatrix} \begin{bmatrix} 0 \\ h_1 - h_P \\ y_1 \end{bmatrix} \tag{37}$$

However, the vector (X_v, Y_v, Z_v) is collinear with the normal vector of the isocurvature surface at point P; therefore

$$|c_P| = \frac{|Z_v|}{\sqrt{X_v^{\,2} + Y_v^{\,2}}} \tag{38}$$

From Transformation (37), the absolute value of the parameter c_P is determined (remember that $\varphi_Q = \varphi_P$) as follows:

$$|c_P| = \frac{|(h_1 - h_P)\cos\phi_P + y_1\sin\phi_P|}{|(h_1 - h_P)\sin\phi_P - y_1\cos\phi_P|} \tag{39}$$

or

$$c_P = \pm\frac{(h_1 - h_P)\cos\phi_P + y_1\sin\phi_P}{(h_1 - h_P)\sin\phi_P - y_1\cos\phi_P} \tag{40}$$

This formula is not valid at points where the normal vector of the isocurvature surface is parallel to the Z–axis because in that case c_P becomes infinite. Hence, the points are excluded at which it holds that

$$\frac{y_1}{(h_1 - h_P)} = \tan\phi_P \tag{41}$$

From Eq. (40) c_P is determined and then the partial derivative k_y is determined from Eq. (34). The final relation is as follows:

$$\begin{aligned} k_y = \frac{c_P\cos\phi_P - \sin\phi_P}{\cos\phi_P + c_P\sin\phi_P} \Bigg\{ & \frac{2(\omega^2 + \gamma J)}{[\gamma - 2(\omega^2 + \gamma J)h_P]^2} \left\{ k + \left[k(k_1 - k_2) + 2\frac{\partial J}{\partial y} \right] h_P \right\} + \\ & + \frac{\gamma}{[\gamma - 2(\omega^2 + \gamma J)h_P]} \left[k(k_1 - k_2) + 2\frac{\partial J}{\partial y} \right] \Bigg\} \end{aligned} \tag{42}$$

Finally, it is worth mentioning that if c_P is equal to zero, then the normal vector to the isocurvature surface at point P is parallel to the meridian plane (see Relations (34) and (29)). In this case Relation (42) becomes

$$\begin{aligned} k_y = -\tan\phi_P \Bigg\{ & \frac{2(\omega^2 + \gamma J)}{[\gamma - 2(\omega^2 + \gamma J)h_P]^2} \left\{ k + \left[k(k_1 - k_2) + 2\frac{\partial J}{\partial y} \right] h_P \right\} + \\ & + \frac{\gamma}{[\gamma - 2(\omega^2 + \gamma J)h_P]} \left[k(k_1 - k_2) + 2\frac{\partial J}{\partial y} \right] \Bigg\} \end{aligned} \tag{43}$$

V. CONCLUSIONS

The sequence of steps required to study the isocurvature surfaces of the normal gravity field were outlined. Along isocurvature surfaces, the normal plumb line curvature has a constant value. The study of these surfaces is feasible only in small coordinate patches and for points not too far away from the ellipsoid (points on Earth's physical surface). That is because

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the function of the plumb line curvature is itself a very complicated function. The vector equation of an isocurvature surface is characteristic. Two variables were used, one angular (geodetic longitude) and one linear. The geodetic latitude serves only as a parameter in the vector equation and not as a variable. The advantage of this parameterization is that the parametric lines of the isocurvature surface are also lines of curvature.

One global significant geometrical property of isocurvature surfaces is that, as the plumb line curvature tends to zero, they tend asymptotically to the equator's plane and to the *Z*-axis. In addition, the normal vector of an isocurvature surface is collinear with the vector *gradk*. Finally, a method for the determination of the surface derivative k_y of a plumb line's curvature was described without differentiating the very complicated function of a plumb line's curvature. This theoretical application shows the usefulness of isocurvature surfaces. The study of isocurvature surfaces may reveal new unknown geometrical properties of the normal gravity field.

REFERENCES

[1] A. Marussi, "Intrinsic Geodesy," WADC Technical Report 52-149, Ohio Sate University Research Foundation, pp. 1-91, 1952.

[2] F. Sanso, "Molodensky's problem in gravity space: A review of the first results," *Journal of Geodesy*, vol. 52, no. 1, pp. 59-70, 1978.

[3] G. Manoussakis and D. Delikaraoglou, "On the gradient of curvature of the plumblines of the Earth's normal gravity field and its isocurvature lines," *Studia Geophysica & Geodaetica*, vol. 55, no. 3, pp. 500-514, 2011.

[4] B. Hofmann-Wellenhof and H. Moritz, *Physucal Geodesy*, second edition, Springer, Wien, New York, p. 72, 2006.

[5] G. Manoussakis, D. Delikaraoglou, and G. Ferentinos, "An alternative approach for the determination of orthometric heights using a circular – arc approximation for the plumbline," IAG vol. 133, Commission 2, "GS002 Symposium for the Gravity Field," General Assembly of the International Union of Geodesy and Geophysics, Perugia, Italy, July 2-13, 2007, Springer Verlag, 2008.

[6] G. Manoussakis, "Estimation of the normal Eötvös matrix for low geometric heights," *Acta Geodaetica e Geophysica Hungarica*, vol. 48, no. 2, pp.179-189, June 2013.

[7] W. Torgue, *Geodesy*, second ediion, Walter de Gruyter, Berlin, New York, p. 57, 1991.

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