Non-Proportionality of Probability Weighting Function in Prospect Theory

Identification of Non-Proportionality

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***Abstract*-In prospect theory, the non-proportionality of probability weighting function $\pi(p)$ is assumed. However, this belief is based on the results of a questionnaire survey on the change of risk attitude, which used two extreme probabilities (0.9 and 0.002) of being rewarded. This paper investigated how the probabilities between the two extremes affect decision-making, as this has not been explored previously. An attempt was made to identify the condition of non-proportional property in prospect theory. It was discovered that the *p*-value differed among the participants when the transition of choice from more certain decision to less certain decision occurred. The *p*-value was dispersive to a larger extent, and ranged from 0.001% to 70%. This showed that the assumption on non-proportionality in prospect theory is not necessarily universal, and the condition of non-proportionality is ruled, to a large extent, by individual difference.**

Keywords- Prospect Theory; Irrationality; Decision Making; Lower Probability of Occurrence; Moderate Probability of Occurrence; Non-Proportionality

I. INTRODUCTION

In prospect theory [1-8], the non-proportionality of probability weighting function $\pi(p)$ is assumed. However, this belief was based on a questionnaire on the change of risk attitude, which used two extreme probabilities (0.9 and 0.002) of being rewarded.

In the study of Kahneman and Tversky [5], the condition under which non-proportionality holds was not discussed systematically. In other words, how the probabilities between two extremes (p=0.9 and 0.002) affect the decision-making on two prospects A=($6,000, p/2) and B=($6,000, p) has not been explored. Moreover, it is unclear when the transition from “more certain” to “less certain” occurred whether the *p*-value is approximately the same among the participants. It seems more reasonable to assume that the *p*-value might be dispersive to a large extent. It is not clear whether the assumption on non-proportionality in prospect theory is universal. It is also not certain to what extent the condition of non-proportionality is ruled by individual difference.

Therefore, how the value of p affects the non-proportional property in decision-making on risk attitude was investigated in order to clarify whether non-proportionality holds universally or under specific conditions. It was also examined under what condition the reversal of choice was observed. It should be noted that the condition other than r was the same as that of Kahneman and Tversky. In summary, how the probabilities between the two extremes affect the decision making were investigated, and an attempt was made to identify the condition that the non-proportional property in prospect theory holds.

II. NON-PROPORTIONALITY IN PROSPECT THEORY

In prospect theory, the following non-proportionality of probability weighting function $\pi(p)$ is assumed [8]. The variable p corresponds to an actual probability, and q and r are variables that satisfy $0 \leq q, r \leq 1$.

$$\frac{\pi(pqr)}{\pi(pr)} > \frac{\pi(pq)}{\pi(p)} \tag{1}$$

Kahneman and Tversky, using the following prospects on risk attitude and for a group of 66 participants, discussed non-proportionality in prospect theory:

Problem 1

(A) Get $6,000 with the probability of 0.45 (prospect A = ($6,000, 0.45))

(B) Get $3,000 with the probability of 0.9 (prospect B = ($3,000, 0.9))

Problem 2

(C) Get $6,000 with the probability of 0.001 (prospect C = ($6,000, 0.001))

(D) Get $3,000 with the probability of 0.002 (prospect D = ($3,000, 0.002))

In Problem 1, 14% and 86% of the participants chose (A) and (B), respectively. In Problem 2, 73% and 27% of the participants chose (C) and (D), respectively. The two prospects in Problem 1 are generally expressed as follow:

Prospect A: Get $*y* with the probability of *pq*.

Prospect B: Get $*x* with the probability of *p*.

The two prospects in Problem 2 are generally expressed as follow:

Prospect C: Get $*y* with the probability of *pqr*.

Prospect D: Get $*x* with the probability of *pr*.

For p=0.9, q=0.5, x=3000, y=6000, and r=1/450, prospects B and C were chosen, and thus the reflection effect was observed. This can be expressed as follows:

$$\pi(pq)v(y) < \pi(p)v(x) \tag{2}$$

$$\pi(pq\mathrm{r})v(y) > \pi(p\mathrm{r})v(x) \tag{3}$$

Here, $v(x)$ corresponds to the value function in prospect theory. Eq. (2) and Eq. (3) can be expressed as follow:

$$\frac{\pi(pq)}{\pi(p)} < \frac{v(x)}{v(y)}$$

$$\frac{\pi(pqr)}{\pi(pr)} > \frac{v(x)}{v(y)}$$

Therefore, Eq. (1) holds.

In the study of Kahneman and Tversky, the condition under which Eq. (4) holds was not discussed systematically. Therefore, how the value of r affects the non-proportional property in decision-making on choice was explored in order to clarify whether non-proportionality holds universally or merely under a specific condition. It was also examined under what condition the reversal of choice was observed. It should be noted that the condition other than r was the same as that of Kahneman and Tversky.

III. METHOD

A. Participants

Twenty-one graduate and undergraduate students aged 20 to 26 years took part in the experiment. None had any knowledge or skill in psychology or behavioral economics. Consent was given by all participants after they received a brief explanation of the aim and contents of the experiment.

B. Task

First, the participant was required to answer Problem 1, which is provided below.

Problem 1

Which would you choose?

(A) Get $6,000 with the probability of 0.45.

(B) Get $3,000 with the probability of 0.9.

The participant who chose (B) was required to proceed to Problem 2-*p* below:

Problem 2-p

(A) Get $6,000 with the probability of p/2.

(B) Get $3,000 with the probability of p.

In such a way, the experiment was repeated until the participant chose alternative (A). When the participant chose (A), the experiment ended. The following sixteen probabilities p of getting $3,000 were used in the experiment: 70%, 50%, 30%, 10%, 5%, 4%, 2%, 1%, 0.5%, 0.4%, 0.3%, 0.2%, 0.1%, 0.01%, 0.001%, and 0.0001%. In this experiment, the order of judgment for fifteen values of p was not randomized, because it was possible that multiple reversals of choice from (B) to (A) occur if random presentation of p was adopted.

C. Procedure

The study of Kahneman and Tversky [5], on the basis of Problem 1 and Problem 2, obtained the characteristics of non-proportionality. As 86% of the participants chose (B) in Problem 1, it was assumed that the following inequality generally holds. The values of p, q, and r were p=9/10, q=1/2, and r=1/450.

$$\pi(0.9)v(\$3000) > \pi(0.45)v(\$6000)$$
$$\therefore \frac{v(\$3000)}{v(\$6000)} > \frac{\pi(0.45)}{\pi(0.9)} \tag{4}$$

As alternative (C) was chosen by the majority of participants (73%) in Problem 2, it was assumed that the following inequality holds:

$$\pi(0.001)v(\$6000) > \pi(0.002)v(\$3000)$$
$$\therefore \frac{\pi(0.001)}{\pi(0.002)} > \frac{v(\$3000)}{v(\$6000)} \tag{5}$$

From Eqs. (4) and (5), the following inequalities hold:

$$\therefore \frac{\pi(0.001)}{\pi(0.002)} > \frac{v(\$3000)}{v(\$6000)} > \frac{\pi(0.45)}{\pi(0.9)} \tag{6}$$

This might be generalized as Eq. (1). In this study, how r (or p) affects the reversal of choice was explored; that is, the change of choice from alternative (B) (Get \$3,000 with the probability of p) to alternative (A) (Get \$6,000 with the probability of p/2). Each participant was tested separately to avoid conformity to the answers of other participants.

On the basis of such an investigation, it was explored how the value of r affects the non-proportional property in decision-making on risk attitude in order to clarify whether non-proportionality holds universally or under a specific condition.

IV. RESULTS

For Problem 1, 20 out of 21 participants (95.2%) chose (B) (Get \$3,000 with the probability of 0.9). The mean value of p when the reversal of choice occurred for these 20 participants was 0.2253 (22.53%). The frequency distribution of p-value for the 20 participants is shown in Table 1. Here, p=90/(1/r). On average, the participants appeared to choose (A) from the following alternatives:

(A) Get \$6,000 with the probability of 0.11265.

(B) Get \$3,000 with the probability of 0.2253.

TABLE 1 FREQUENCY OF CHOSEN *P*-VALUE (1/*R*-VALUE)

p- value	70%	50%	30%	10%	5%	0.5%	0.1%
1/r- value	1.286	1.8	3	9	18	180	900
frequency	1	5	2	5	4	1	2

Mean *p* -value: 22.53% S.D. of *p* -value: 22.53%

V. DISCUSSION

On average, the following choices were conducted by the 20 participants who chose alternative (B) in Problem 1:

$$\pi(0.9)v(\$3000) > \pi(0.45)v(\$6000)$$
$$\therefore \frac{v(\$3000)}{v(\$6000)} > \frac{\pi(0.45)}{\pi(0.9)} \tag{7}$$

When the choice changed from (B) to (A), the following relationship, on average, held:

$$\pi(0.11265)v(\$6000) > \pi(0.2253)v(\$3000)$$
$$\therefore \frac{\pi(0.11265)}{\pi(0.2253)} > \frac{v(\$3000)}{v(\$6000)} \tag{8}$$

In Eq. (8), the mean *p*-value 22.53% of 20 participants was used. From Eqs. (7) and (8), the following inequality (non-proportionality) held:

$$\frac{\pi(0.11265)}{\pi(0.2253)} > \frac{v(\$3000)}{v(\$6000)} > \frac{\pi(0.45)}{\pi(0.9)} \tag{9}$$

By making p=0.9, q=1/2, and r=1/3.995, Eq. (9) can be expressed as Eq. (1). It must be noted that this relationship was obtained for the mean value of p. The p value when the transition of choice from (B) to (A) occurred differed among the participants. As shown in Table 1, the *p*-value was dispersive to a large extent, and ranged from 0.1% to 70%. This shows that the assumption on non-proportionality in prospect theory is not necessarily a universal one, and the condition of non-proportionality (Eq. (1)) is ruled by individual differences to a large extent.

For the 5 participants in Table 1 (p=0.1(10%)), the situation of decision-making can be summarized as follows:

Problem 1

Which would you choose?

(A) Get $6,000 with the probability of 0.45.

(B) Get $3,000 with the probability of 0.9.

Alternative (B) was selected.

Problem 2-0.1

Which would you choose?

(A) Get $6,000 with the probability of 0.05.

(B) Get $3,000 with the probability of 0.1.

The choice changed from (B) to (A) when p was equal to 0.1(10%). When p was less than 0.1(10%), the property of non-proportionality was true for these participants. For the two participants (see Table 1), their choice changed from (B) to (A) when the *p*-value was 0.001(0.1%).

For almost all of the participants (20 out 21 participants), the change of choice from (B) (Get $3,000 with the probability of p) to (A) (Get $6,000 with the probability of $p/2$) was observed for the *p*-value peculiar to each participant. Generally, the *p*-value ranged from 0.1% to 70%. Thus, it was verified that the non-proportionality held for less than the identified *p*-value, although the *p*-value differed significantly among the participants.

Next, the comparative discussion between this study and Kahneman and Tversky [5] was conducted. Fourteen percent of the participants chose (A) (86% chose (B)) in the study of Kahneman and Tversky [5], while 4.76% of the participants in this study chose (A) (95.24% chose (B)). In the range of this comparison, Japanese participants seem to be more risk aversive.

Although the experimental procedure differed from this study, 73% of the participants in the Kahneman and Tversky study [5] chose (A) (Get $6,000 with the probability of 0.1%) when p reached 0.2%. The corresponding *r*-value was 1/450. On the other hand, as shown in Table 1, the *p*-value of only two participants (10% (=2/20)) was less than 0.2%. On average, the *p*-value in this study was 22.35% (r=1/4.027) and by far larger than the *p*-value of 0.02% used in the study of Kahneman and Tversky.

The participants in the Kahneman and Tversky study [5] were forced to choose alternatives (A) or (B) when p was 0.2% in order to obtain the ratio of participants whose choices changed from (B) (Get $3,000 with the probability of 0.2%) to (A) (Get $6,000 with the probability of 0.1%). However, this study investigated the *p*-value when the change of choice from (A) to (B) occurred. It could be speculated that the participants in the study of Kahneman and Tversky [5] were less susceptible to the *p*-value for the occurrence of change of choice than the Japanese participants in this study. The participants in this study appeared to make their choice change for larger *p*-values than those in the Kahneman and Tversky study [5].

The results were discussed using a probability weighting function, as shown in Figs. 1(a) and 1(b). Here, two types of probability weighting functions were assumed. While the probability weighting function of Fig. 1(a) consists of concave and convex curves, the function in Fig. 1(b) consists of only a convex curve. When the reversal of choice from (B) to (A) did not occur in Problem 2-p, the difference between $\pi(p_1)$ and $\pi(p_1/2)$ was larger compared to the difference between $\pi(p_2)$ and $\pi(p_2/2)$, as shown in Figs. 1(a) and 1(b). The decision was made by synthesizing the probability weighting function and the value function. However, it is difficult to separate the effects of the probability weighting function and the value function on the decision. Although one cannot be certain that the smaller difference between the values of probability weighting functions at p_2 and $p_2/2$ was the dominant cause of the reversal of choice from (B) to (A) in Problem 2-p, it can be speculated that the smaller difference between $\pi(p_2)$ and $\pi(p_2/2)$ contributed to the reversal of choice to some extent. As this affected the decision in Problem 2-p by increasing the total evaluation (utility) of option (A), the reversal of choice must have occurred at p_2. In such a way, it was demonstrated that the reversal of choice from (B) to (A) in Problem 2-p occurred when the probability weighting function was shaped like Figs. 1(a) and 1(b) and satisfied the condition that the difference between the values of probability weighting functions at p_2 and $p_2/2$ got smaller compared to the difference between those at p_1 and $p_1/2$. According to the probability weighting function in Figs. 1(a) and 1(b), the non-proportionality of the probability weighting function (Eq. (1)) held at the probability when the reversal of choice from (B) to (A) occurred.

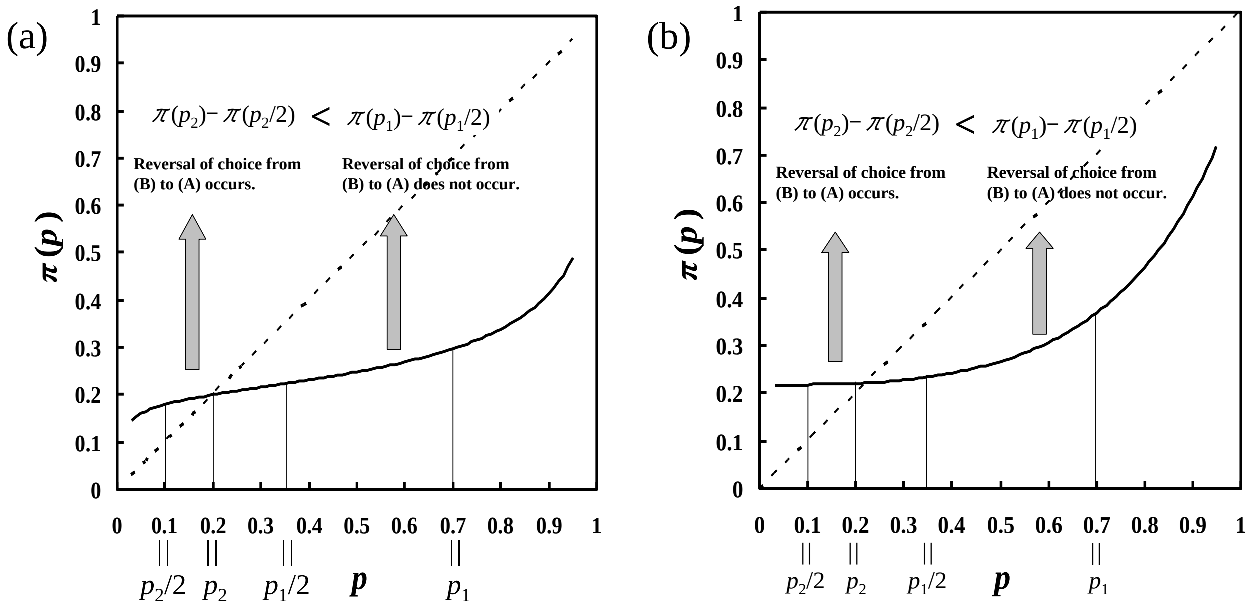

Fig. 1 Explanation of reversal of choice from (B) to (A) using a probability weighting function. It was assumed that the reversal of choice occurred at p_2

In this study, the expected values of alternatives (A) and (B) were the same for both Problem 1 and Problem 2-p. Future research must explore the cases where the expected values of both alternatives are not the same so that the non-proportional property of the probability weighting function can be more generalized. Future research should also generalize the results further using the following decision-making problem on risk attitude where the effect of not only p, but also X and Y on the risk attitude, must be identified.

Which would you choose?

(A) Get \$$X$ with the probability of $p/2$.

(B) Get \$$Y$ with the probability of p.

VI. CONCLUSIONS

Because how the probabilities between two extremes affect decision-making had not yet been explored, this study aimed to do that, and an attempt was made to identify the condition of non-proportional property in prospect theory.

The p-value when the transition of choice from (B) to (A) occurred differed among the participants. The p-value was dispersive to a large extent, and ranged from 0.1% to 70%. This showed that the assumption on non-proportionality in prospect theory is not necessarily a universal one, and the condition of non-proportionality (Eq. (1)) is affected by individual differences for the most part.

Future research should increase the number of participants and verify the validity of the study.

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