Water Resources Planning and Management by Use of Generalized Benders Decomposition Method to Solve Large-scale MINLP Problems

André A. Keller

Computer Science Laboratory (LIFL) / SMAC Unit, Lille 1 University – Science and Technology
Cité Scientifique, 59650 Villeneuve- d'Ascq, France

Andre.Keller@univ-lille1.fr

***Abstract*-Water resources systems (WRS) models involve a large number of continuous and integer quantities. Water quality management problems also require the consideration of uncertainties related to the variability of flow streams and temperatures. WRS dynamics are primarily nonlinear. These characteristic features suggest the use of (stochastic) mixed-integer programming models, as well the use of sensibility analysis and simulations. Moreover, high dimensional real-world models and combinatorial alternatives require adequate tools for large-scale optimization models. These techniques consist of decomposition methods such as the generalized Benders decomposition (GBD) and the branch-and-bound enumerative algorithm. This contribution introduces the subject of modeling WRSs by use of GBD and branch-and-bound algorithms with numerical applications.**

Keywords- Mixed-Integer Nonlinear Programming; Large-Scale Optimization Methods; Generalized Benders Decomposition Algorithm; Water Resource System; Groundwater Management; Water Quality Management

I. INTRODUCTION

This contribution[1] introduces water resources problems that involve continuous and discrete quantities. For example, in groundwater management systems, pumping rates are continuous parameters, while discrete parameters include the locations of wells. Water quality control problems are related to additional uncertainties, which are notably due to natural variability in stream flows and temperatures, as well as random pollution transport. These characteristics determine the type of mathematical program to be used, such as mixed-integer programming (MIP) and stochastic MIP.

Numerous simulation and optimization models[1] have been proposed in practice. Some of these models formalize the interactions between surface water and sub-surface flow systems [2]. Optimization models correspond to linear and nonlinear programming, dynamic programming, and stochastic programming. Real-life water resources systems (WRS) involve two additional complications: the large-size models with thousands of decision variables and equations, and the combinatorial number of alternative solutions. Specific techniques have been developed to combat these two challenges.

The first challenge is how to best take into account the high dimensional WRSs. Dantzig-Wolfe [3] initiated an extensive study of large-scale programming methods [4]. The procedure consists of a master program with interacting sub-problems. Sub-problems receive a set of simplex multipliers, and apply their solution to the master program. The master program optimally combines this information to compute new duals. This procedure was applied to block-angular structures in which blocks are linked by coupling equations. The second challenge is to reduce the number of alternatives to be examined. The branch-and-bound enumeration technique was proposed by Land & Doig (1960) [5], and consists of two basic operations. The first operation is a branching step which partitions the feasible space into smaller subsets. The second operation is a bounding step to calculate a lower bound (for a minimization problem) within each subset ([6-8].

This article is organized as follows: Section II formulates general and particular mixed-integer nonlinear programming (MINLP) problems. Section III introduces the decomposition methods, focused on the GBD algorithm for which a numerical example illustrates all iteration steps necessary to obtain an optimal solution (Appendix A). Section IV characterizes WRS applications from previous literature; these applications include WRS planning and management problems, water management problems, and water quality management problems. Appendix A offers an illustrative numerical example of the GBD algorithm. Appendix B solves a simple groundwater supply system serving industry.

II. STRUCTURE OF MINLP SYSTEMS

An MINLP problem utilizes a general formulation in which the discrete decision variables are integers. We demonstrate that the integrality constraints can be expressed by means of binary variables. Bloc-separable formulations are efficient for

[1] Simulation and optimization models are different approaches. Simulation models aim to approximate the system behaviour taking all its characteristics. They may include experience and judgment of the planners and designers. They can provide helpful responses of the system to some decision rules. Today, preference would be to incorporate optimization routines in a simulation model. A linked simulation-optimization model application is in Finney et al. (1977) [1].

DOI: 10.5963/JWRHE0402003

large-scale systems. In this context, a Lagrangian relaxation yields partial Lagrangian sub-problems.

A. MINLP Formulation with Integer Variables

The general formulation of an MINLP problem is as follows:

$$\begin{aligned} \underset{(\mathbf{x},\mathbf{y})\in X\times Y}{\text{minimize}} \quad & f(\mathbf{x},\mathbf{y}) \\ \text{subject to:} \quad & \mathbf{g}(\mathbf{x},\mathbf{y}) \le \mathbf{0}, \\ & \mathbf{h}(\mathbf{x},\mathbf{y}) = \mathbf{0}, \\ & \mathbf{x} \in X \subseteq \mathbf{R}^n,\ \mathbf{y} \in Y \text{ integer}, \end{aligned} \tag{1}$$

where $f : X \times Y \mapsto \mathbf{R}$, $\mathbf{g} : X \times Y \mapsto \mathbf{R}^m$, and $\mathbf{h} : X \times Y \mapsto \mathbf{R}^p$. The conditions for which the Equation (1) has a finite optimal solution and optimal multipliers were previously noted by Floudas (1995) [9], pp. 114-115.

B. MINLP Formulation with Binary 0-1 Variables

An integrality constraint such as $y \in \mathbf{Z} \cap \left[y^L, y^U \right]$ can be expressed by N binary 0-1 variables. In fact, $y = y^L + z_1 + 2^1 z_2 + 2^2 z_3 + \cdots + 2^{N-1} z_N$, where N denotes the minimum necessary number of 0-1 variables. Number N is determined by $N = 1 + \operatorname{int}\left(\dfrac{\ln\left(y^U - y^L\right)}{\ln(2)} \right)$, where $\operatorname{int}(.)$ truncates its real argument to an integer value. Then, $\mathbf{y} \in Y = \{0,1\}^q$. Suppose that $X \in \{1,2,3\}$. Using k 0-1 variables, the maximum value can be represented by $3 = 2^k - 1$ then we find $k = 2$. We then obtain $y = 1 + z_1 + 2z_2$, where z_1 and z_2 are 0-1 variables.

C. Block-separable MINLP Formulation

Let an NLP problem (without equalities) be represented by separable functions, and suppose that there exists a partition $\{J_1, \ldots, J_p\}$ of set $\{1, \ldots, n\}$. We have $\bigcup_{k=1}^{p} J_k = \{1, \ldots, n\}$ and $J_i \cap J_j = \varnothing$ for all $i \neq j$.

$$\begin{aligned} \underset{\mathbf{x}\in X}{\text{minimize}} \quad & f(\mathbf{x}) = \sum_{k=1}^{p} f_k\left(\mathbf{x}_{J_k}\right) \\ \text{subject to:} \quad & g_j(\mathbf{x}) = \sum_{k=1}^{p} g_{jk}\left(\mathbf{x}_{J_k}\right) \le 0,\ j = 1,\ldots,m \\ & X = \left\{ \mathbf{x} \in \mathbf{R}^n : \mathbf{x}_{J_k} \in X_k,\ k = 1,\ldots,p \right\}. \end{aligned} \tag{1}$$

$\mathbf{x}_{J_k}$ denotes a sub-vector of $\mathbf{x}$, such as we get $\mathbf{x} = \mathbf{x}_{J_1} \| \mathbf{x}_{J_2} \| \ldots \| \mathbf{x}_{J_p}$, after index reordering. The functions in Equation (1) $f, g_j : \mathbf{R}^n \mapsto \mathbf{R}$ and the set $X \subset \mathbf{R}^n$ are block separable. In this context, a Lagrangian relaxation decomposes into p partial Lagrangian problems, as with Equation (2) (Nowak, 2005 [10]).

$$\inf_{\mathbf{x}_{J_k} \in X_k} \left\{ f_k\left(\mathbf{x}_{J_k}\right) + \sum_{j=1}^{m} \mu_j g_{jk}\left(\mathbf{x}_{J_k}\right) \right\},\ k = 1,\ldots,p. \tag{2}$$

The particular structure of MINLP problems corresponds to a formulation in which nonlinearities only concern continuous variables, while discrete variables appear separably and linearly. We may write:

DOI: 10.5963/JWRHE0402003

$$\begin{aligned} &\underset{\mathbf{x}\in X,\mathbf{y}\in Y}{\text{minimize}} && \mathbf{c}^T\mathbf{y} + f\left(\mathbf{x}\right) \\ &\text{subject to :} && \mathbf{g}\left(\mathbf{x}\right) + \mathbf{A}\mathbf{y} \le \mathbf{0}, \\ & && \mathbf{h}\left(\mathbf{x}\right) = \mathbf{0}, \end{aligned} \tag{3}$$

where $X = \left\{\mathbf{x} \in \mathbf{R}^n : \mathbf{x} \in \left[\mathbf{x}^L, \mathbf{x}^U\right]\right\}$ and $Y = \left\{\mathbf{y} \in \mathbf{Z}^q : \mathbf{y} \in \left[\mathbf{y}^L, \mathbf{y}^U\right], \mathbf{B}\mathbf{y} \le \mathbf{d}\right\}$.

III. GENERALIZED BENDERS DECOMPOSITION

Mixed-integer linear programming (MILP) problems can be solved by using inner or outer approximations, or integrated decomposition methods. Decomposition methods solve large-scale systems in operations research. This section is dedicated to the GBD algorithm, which we present with a complete numerical example (Appendix A).

A. Decomposition Methods

The decomposition method used to solve MILP consists of generating approximations to the convex hull of a polyhedral feasible set. Traditional methods are inner approximation (e.g., Danzig-Wolfe method, Lagrangian method) and outer approximation[2] (e.g., cutting-plane method, relaxation algorithms). Other integrated decomposition methods include the price-and-cut and relax-and-cut methods [4].

Decompositions methods solve large-scale problems in operations research by exploiting the structure of programming problems. The principle is the splicing of difficult problems into smaller sub-problems which involve complicated constraints. These problems are coupled with a simple master problem in high dimension. Nowak (2005) [10] considers four decomposition principles including the Lagrange decomposition method, primal cutting-plane method, column generation and Benders decomposition. Floudas (1995) [9] describes the methods best suited to solving MINLP problems; he considers notably generalized Benders decomposition (GBD), branch-and-bound (BB), outer approximation (OA), and generalized cross decomposition (GCD).

B. GBD Algorithm

The GBD algorithm generates an upper bound and a lower bound of the approximated solution at each iteration point. The original problem with continuous $\mathbf{x}$ variables and discrete $\mathbf{y}$ variables is decomposed into two problems: a primal sub-problem, and a master problem. The primal sub-problem corresponds to the original problem with fixed discrete variables; his problem provides information about the upper bound and Lagrange multipliers in the $\mathbf{x}$-space. The master problem is derived from nonlinear duality theory and provides information about the lower bound and calculates the next set of values for the discrete $\mathbf{y}$ variables. The sequence of updated upper bounds is nonincreasing, and the sequence of lower bounds is nondecreasing. A convergence is attained after a finite number of iterations (Floudas, 1995).

The GBD algorithm[3] consists of an initialization step and two additional steps for each iteration. Let us consider a simplified MINLP problem $\mathcal{P}$ in Equation (5), with one continuous decision variable $x \in \mathbf{R}$ and one integer decision variable $y \in \mathbf{Z}$. The functions f, $g_i\left(i = 1,\ldots,m\right) : X \times Y \mapsto \mathbf{R}$ are convex and differentiable, X is a compact set and Y is a finite integer set.

$$\mathcal{P} = \underset{x,y}{\text{minimize}} \left\{ f\left(x,y\right) \text{ s.t. } g_i\left(x,y\right) \le 0,\, i = 1,\ldots,m;\, x \in X \subset \mathbf{R}, y \in Y \subset \mathbf{Z}\right\}, \tag{4}$$

The GBD algorithm[4] consists of the following steps:

- **Step 0**: Choose $y^1 \in Y$. Set index set $I^0 = J^0 = \varnothing$ and iteration counter to $k = 1$. Then, set bounds to $LB^0 = UB^0 = -\infty$.
- **Step 1:** Solve $\mathcal{P}\left(y^k\right)$. Two cases are described below.

[2] Recall that inner approximation solves a sequence of approximations whose feasible regions are included in the original feasible region. On the contrary, an outer approximation includes the entire initial feasible region.
[3] See Galati (2010)[11] for a complete presentation with numerical examples.
[4] The original algorithm is due to Benders(1962)[12]. Benders' approach was extended by Geoffrion (1972)[13] to a broader class of mathematical programs.

DOI: 10.5963/JWRHE0402003

►If $\mathcal{P}\left(y^{k}\right)$ is feasible: an optimal solution is $\hat{x}^{k}$, and optimal duals are $\boldsymbol{\lambda}^{k} \in \mathbf{R}^{m}$. Set index determines $I^{k} = I^{k-1} \cup \{k\}$ and $J^{k} = J^{k-1}$. Set upper bound $UB^{k} = \min\left\{ f\left(x^{k}, y^{k}\right), UB^{k-1} \right\}$. If $UB^{k} = f\left(x^{k}, y^{k}\right)$, then STOP and obtain the optimal solution $(\hat{x}, \hat{y}) = (x^{k}, y^{k})$.

►If $\mathcal{P}\left(y^{k}\right)$ is infeasible: solve an equivalent ℓ_{∞}-minimization of constraint violations[5] (Equation (6)).

$$\mathcal{Q}\left(y^{k}\right) = \underset{x \in X}{\text{minimize}} \left\{ \beta \text{ s.t. } g_{i}\left(x, y^{k}\right) \le \beta,\, i = 1, \ldots, m \right\} \tag{5}$$

Obtain an optimal solution $\hat{x}^{k}$ and optimal duals $\hat{\boldsymbol{\mu}}^{k} \in \mathbf{R}^{m}$. Set $J^{k} = J^{k-1} \cup \{k\}$ and $I^{k} = I^{k-1}$.

- Step 2: Solve a master problem[6].

$$\begin{aligned} &\text{minimize} \quad \alpha \\ &\text{subject to:} \quad \underset{x \in X}{\text{minimize}} \;\; L(x, y, \boldsymbol{\lambda}) \le \alpha,\ \forall \boldsymbol{\lambda} \ge \mathbf{0}, \\ &\qquad\qquad\quad \underset{x \in X}{\text{minimize}} \;\; \boldsymbol{\mu}^{T} \mathbf{g}(x, y) \le 0,\ \forall \boldsymbol{\mu} \ge \Lambda, \\ &\qquad\qquad\quad y \in Y, \end{aligned} \tag{6}$$

where $L(x, y, \boldsymbol{\lambda}) = f(x, y) + \boldsymbol{\lambda}^{T}\mathbf{g}(x, y)$ and $\Lambda = \left\{ \sum_{i=1}^{m} \mu_{i} = 1,\ \mu_{i} \ge 0,\ i = 1, \ldots, m \right\}$. Using a relaxation for this master problem, solve Equation (8).

$$\begin{aligned} &\text{minimize} \quad \alpha \\ &\text{subject to:} \;\; L\left(x^{i}, y^{i}, \boldsymbol{\lambda}^{i}\right) + \nabla_{y} L\left(x^{i}, y^{i}, \boldsymbol{\lambda}^{i}\right) \times \left(y - y^{i}\right) \le \alpha,\ i \in I^{k}, \\ &\qquad\qquad \left(\boldsymbol{\mu}^{j}\right)^{T} . \left(\mathbf{g}\left(x^{j}, y^{j}\right) + \nabla_{y} \mathbf{g}\left(x^{j}, y^{j}\right) \times \left(y - y^{j}\right)\right) \le 0,\ j \in J^{k}. \end{aligned} \tag{7}$$

Set $LB^{k} = \alpha^{k}$. If $LB^{k} \ge UB^{k}$, then STOP and obtain the optimal solution $(\hat{x}, \hat{y}) = (x^{k}, y^{k})$. Otherwise, set $k = k + 1$ and return to Step 1.

IV. SOLVING WATER RESOURCE PLANNING AND MANAGEMENT PROBLEM

This section addresses surveys concerning the planning and management of water using mathematical programming methods. A small-size model applied to groundwater management illustrates the mathematical program. A comparison of recent applications in previous literature addresses WRS, groundwater and water quality management.

A. Water Resources Planning and Management Problems and Techniques

Numerous review articles from previous literature focus on mathematical techniques and their applications to water resources planning [15-17]. Recently, Husain [18] surveyed optimization models that are applicable to reservoir operation problems such as linear programming, dynamic programming, and fuzzy logic programming. These applications are also utilized in the surveys by Simonovic [19] and Wurbs [20]. Bragalli et al. [21] extended the applications to design problems of a water distribution network.

---

[5] The minimization problem can be expressed by $\text{minimize} \left\| \mathbf{g}\left(x, y^{k}\right) \right\|_{\infty}$, where the norm is defined by $\left\| \mathbf{g}\left(x, y^{k}\right) \right\|_{\infty} = \max\left\{ g_{1}\left(x, y^{k}\right), \ldots, g_{m}\left(x, y^{k}\right) \right\}$.

[6] This master problem in which the constraints are value functions, is deduced from the Lagrangian dual of $\mathcal{Q}(y)$ (see Li & Sun (2006) [14], p.379).

DOI: 10.5963/JWRHE0402003

B. Modeling a Groundwater Planning Problem

The following illustrative model is drawn from Kwanyuen and Fontane [22]. This model determines optimal well locations, as well as timing and pumping rates subject to economic and technical constraints. The demand is satisfied, and the construction and pumping operation costs are minimized.

The following variables are in alphabetical order:

• $\beta(j,i,k-n+1)$: unit response coefficient describing the effect on the hydraulic head at cell j , in time period k , due to a unit pumping amount at cell i in the current or previous period n; • C_q : construction cost of a well; • C_p : cost of pumping (unit volume per unit head at all wells); • D_{jk} : drawdown due to pumping of well j at period k ; • DL_{jk} : maximum allowable drawdown of well j at time k ;• DM_k : water demand at period k ; • L_j : initial lift at well j ; • M : number of potential pumping wells; • N : number of planning periods; • NP : maximum allowable wells; • Q_{jk} : pumping rate of well j at period k ; • $Q_{jk}^{\max}$: maximum pumping capacity of well j at period k ; • r : interest rate; • X_{jk} : binary 0-1 indicating if well j is constructed at period k ; • Z : total cost of construction and pumping at present value over all the periods.

The MIP problem is described below:

$$\begin{aligned}
&\underset{Q_{jk},X_{jk}}{\text{minimize}} \quad Z=\sum_{j=1}^{M}\sum_{k=1}^{N}\left\{\frac{C_q X_{jk}+C_p\left(L_j+D_{jk}\right)Q_{jk}}{\left(1+r\right)^k}\right\}\\
&\text{subject to: } 1)\ D_{jk}=\sum_{i=1}^{M}\sum_{n=1}^{k}\beta\left(j,i,k-n+1\right)Q_{in},\ \forall j,k\\
&\qquad 2)\ \sum_{j=1}^{M}Q_{jk}\geq DM_k, \forall k\\
&\qquad 3)\ D_{jk}\leq DL_{jk},\ \forall j,k\\
&\qquad 4)\ Q_{jk}^{\max}\left(\sum_{n=1}^{k}X_{jn}\right)-Q_{jk}\geq 0,\ \forall j,k\\
&\qquad 5)\ \sum_{j=1}^{M}\sum_{k=1}^{N}X_{jk}\leq NP,\\
&\qquad 6)\ \sum_{k=1}^{N}X_{jk}\leq 1,\ \forall j.
\end{aligned} \tag{8}$$

The objective function of Equation (8) is to minimize the total discounted cost of development and pumping subject to economic and technical constraints. The continuous decision variable is the pumping rate at each well. The integer decision variables are the location of wells and their time of development. The individual constraints of Equation 9 are described below:

The hydraulic constraint (1) characterizes the relation of pumping to drawdown, which must be satisfied by constraint (2). According to constraint (3), drawdown at each well cannot exceed the maximum allowable drawdown; pumping rate cannot exceed the maximum pumping capacity, and the wells must be constructed before they operate in constraint (4). In constraint (5) the number of wells must be less than or equal to the maximum allowable number of wells. According to constraint (6) a maximum of one well can be developed at each site during the planning period.

This MINLP problem is solved by using a heuristic branch-and-bound method [22].

C. Water Resources Planning and Management Problems and Techniques

The main features of recent applications are displayed in Table 1 and Table 2. Water resources planning and management, groundwater management, and water quality management are typical areas of WRS for which we provide some applications from previous literature. For each application, the main characteristics are: the type of problem, the programming model description (i.e., the decision variables, objectives, and constraints), the methodology, and reference.

DOI: 10.5963/JWRHE0402003

The methods used in WRS planning and management are primarily multi-period networks and cost minimization programs. Decision variables are both continuous and discrete. The cost minimization objective may be included within a weighted set of multiple objectives. The constraints are: balance constraints, the meeting of water demands, and capacities of equipment and plants. MINLP programs are used for groundwater management problems. The GBD algorithm, branch-and-bound method, and other approaches are used to determine solutions. In water quality management problems, stochastic methods are required due to the intrinsic random nature of the data.

TABLE 1 WATER RESOURCE MANAGEMENT APPLICATIONS

#	Type of problem	Programming model description	Type of method	Reference
►WATER RESOURCES SYSTEMS PLANNING & MANAGEMENT				
1	●Planning with shortage control (e.g., dimension of water works, reliability of the system). ●Management (e.g., vulnerability of the system to water resource shortage risk). ●Maintenance of pipe-networks for water supply.	●**Decision variables**: Time independent planning variables (e.g., storage volume of reservoirs, extension of irrigation sites, etc.); time dependent operating variables to meet system requirements. ●**Objectives:** To minimize costs of constructions, maintenance, and operating costs. ●**Constraints**: Mass balance; flow continuity equations; links between planning constraints; filling capacity of reservoirs.	●Multi-period network; optimization model.	Liberatore et al. (2006)[23]
2	●Reservoir system operation for water supply and power generation. ●River basin water allocation. ●Control and salinity.	●**Objectives:** To maximize a weighted combination of objectives: the ratio of delivered water to demand; the smallest water deficit among all sites, the total amount of generated hydropower in the basin; to minimize salt concentration. ●**Constraints**: Water balance at river, at reservoir nodes, at groundwater, etc.; power generation; salinity balance at river, canal.	●Model network; large-scale nonconvex NLP, generalized Bender decomposition algorithm, relaxing the constraints. ●Two models: 1) reservoir operation model; 2) water basin allocation and salinity control model.	Cai et al. (2001)[24]
► GROUNDWATER MANAGEMENT				
3	●Design and water management. ●Least-cost conjunctive use strategies for managing surface water and groundwater resource systems.	●**Decision variables**: Continuous variables (i.e., extraction, injection rates from wells); discrete variables (i.e., well locations). ●**Objectives:** To minimize operation costs. ●**Constraints:** Desired groundwater levels; pumping capacities; meeting water demands.	●MINLP model. ●Solved by branch-and-bound method, GBD algorithm, global search techniques.	Chiu et al. (2012)[2]

TABLE 2 WATER RESOURCE MANAGEMENT APPLICATIONS (FOLLOWED)

#	Type of problem	Programming model description	Type of method	Reference
► GROUNDWATER MANAGEMENT				
4	●Planning and design. ●Large number of alternatives.	●**Decision variables**: continuous variables (i.e., pumping rates at wells); discrete variables (i.e., well locations). ●**Objectives:** To minimize developing and pumping costs. ●**Constraints:** Hydraulic constraints; water demand requirements; maximum allowable drawdown; maximum pumping capacity; maximum number of wells.	●MINLP model by using the response matrix method. ●Solved by notably heuristic branch-and-bound outer approximation, GBD algorithm.	Kwanyuen et al. (1998)[22]
►WATER QUALITY MANAGEMENT				
5	●Determination of adequate treatments to obtain water quality standards at minimum costs.	●**Decision variables**: Continuous variables (i.e., pumping rates); discrete variables (i.e., well locations). ●**Objectives:** To minimize expected total cost. ●**Constraints:** Water quality; state of water quality transition	●Fixed-charge type problem. Stochastic MINLP model. Spatial network of streams in a watershed.	Ali et al. (1998)[25]

DOI: 10.5963/JWRHE0402003

		equations; technology selection and capacities.		
6	●Selection of wastewater treatment facilities in river basin; water quality management.	●**Decision variables**: Discrete waste water treatment levels; random pollution transport; uncertain reaction rates; natural variability in stream flows and temperatures. ●**Objectives:** To maximize the likelihood of management solutions; minimize pollution control costs; maximize water quality; maximize equity. ●**Constraints:** Budget level; water quality goals; random transfer functions involving pollutants.	●MIP model. ●Stochastic branch-and-bound method.	Hägglöf (1996)[26]

V. CONCLUSIONS

Water resources planning and management relates to various problems such as those concerning water surface, groundwater, and water quality requirements. This introductory study demonstrated the benefit of using an MINLP optimization model. Some decision variables (e.g., pumping rates) are continuous, and other variables are discrete (e.g., well locations and installations). High dimensional real-world systems and a combinatorial number of alternatives required large-scale adapted optimization techniques (e.g., GBD and branch-and-bound algorithms). Recent applications prove the benefits of these modeling choices.

This study aimed to produce a unified presentation with completely solved numerical examples, developed by consideration of the intrinsic random nature of data[7] [30]. Further developments could also expand on post-optimization (or sensitivity or parametric) analysis [31, 32], on multi-objective optimization methods, and on fuzzy logic approaches.

ACKNOWLEDGMENT

I would like to thank my referees for their helpful remarks. Further, thanks go to Ziena Optimization Inc. (Canada) for allowing me to use for trials KNITRO™ 9.1 . Software KNITRO for Math Optimization is for finding solutions for both smooth continuous optimization programming problems with constraints, as well as discrete optimization processes with integer or binary variables. In this study, KNITRO solver [33] is notably for solving mixed-integer linear and nonlinear programming problems. This solver uses a branch-and-bound method and the hybrid Quesada-Grossmann approach [35].

REFERENCES

[1] B. A. Finney, W. J. Grenney, A. B. Bishop, and T. C. Hughes, ‘‘Mixed Integer Programming Models for Water Resources Management,’’ Tech. Rep. 492, 1977. Available: http://digitalcommons.usu.edu/water_rep

[2] Y. C. Chiu, T. Nishikawa, P. Martin, ‘‘Hybrid-optimization algorithm for the management of a conjunctive-use project and well field design,’’ Ground Water, vol. 50, no 1, pp. 103-117, 2012.

[3] G. B. Dantzig and P. Wolfe, ‘‘Decomposition principle for linear programs,’’ Oper.Res., vol. 8, no 1, pp. 101-111, 1960.

[4] L. S. Lasdon, Optimization Theory for Large Systems, Mineola, NY, USA: Dover Publications, 2002.

[5] A. H. Land and A. Doig, ‘‘An automatic method for solving discrete programming problems,’’ Econometrica, vol. 28, pp. 497-520, 1960.

[6] E. L. Lawler and D. E.Wood, ‘‘Branch-and bound methods: a survey,’’ Oper.Res., vol. 14, no 4, pp. 699-719, 1966.

[7] L. G. Mitten, ‘‘Branch-and-bound methods: general formulation and properties,’’ Oper. Res. , vol. 18, no 1, pp. 24-34, 1970.

[8] B. Borchers and J. E. Mitchell, ‘‘An improved branch and bound algorithm for mixed integer nonlinear programs,’’ Comput. Oper. Res., vol. 21, no 4, pp. 359-367, 1994.

[9] C. A. Floudas, Nonlinear and Mixed-Integer Optimization: Fundamentals and Applications, Oxford, UK: Oxford University Press, 1995.

[10] I. Nowak, Relaxation and Decomposition Methods for Mixed Integer Nonlinear Programming, Basel, Switzerland: Birkhäuser Verlag, 2005.

[11] M. Galati, ‘‘Decomposition Methods for Integer Linear Programming,’’ Ph.D thesis in Industrial and Systems Engineering, Lehigh University, USA. Available: http://coral.ie.lehigh.edu/~ted/files/paper/MatthewGalatiDissertation09.pdf.

[12] J. F. Benders, ‘’Partitioning procedures for solving mixed-variables programming problems,’’ Numer. Math., vol. 4, pp. 238-252, 1962.

[13] A. M. Geoffrion, ‘’Generalized Benders Decomposition,’’ Optim. Theory Appl., vol. 10, no 4, pp. 237-260, 1972.

[14] D. Li and X. Sun, Nonlinear Integer Programming, New York, USA: Springer, 2006.

[15] W. W. G. Yeh, ‘‘Reservoir management and operations models: a state-of-the art review,’’ Water Resour. Res., vol. no 12, pp. 1797-1818, 1985.

[16] W. W. G. Yeh, ‘‘Systems analysis in ground-water planning and management,’’ J. Water Resour. Plann. Manage., vol. 118, no 2, pp.

[7] We can mention two other aspects such as the use of multi-objective optimization methods in a fuzzy environment [27] and the convexification methods adapted to relaxation procedures[28].

DOI: 10.5963/JWRHE0402003

224-237, 1992.

[17] I. E. Grossmann and Z. Kravanya, ‘‘Mixed-integer nonlinear programming: a survey of algorithms and applications,’’ in L.T. Biegler et al., Eds, Large-Scale Optimization with Applications, part II Optimal Design and Control, New York, USA, Springer, pp. 73-100, 1997.

[18] A. Husain, ‘‘An overview of reservoir systems operation techniques,’’ IJERD, vol.4, no 10, pp. 30-37, 2012.

[19] S. P. Simonovic, ‘’Reservoir systems analysis: closing gap between theory and practice,’’ J. Water Resour. Plann. Manage., vol. 118, no 3, pp. 262-280, 1992.

[20] R.A. Wurbs, ‘‘Reservoir system simulation and optimization models,’’ J. Water Resour. Plann. Manage., vol. 119, no 9, pp.485-472, 1993.

[21] C. Bragalli, C. D’Ambrosio, J. Lee, A. Lodi, and P. Toth, ‘‘An MINLP solution method for a water network’’, in A. Azar and T. Erlebach, Eds, Algorithms- ESA, LNCS 4168, Berlin-Heidelberg, Germany: Springer-Verlag, pp. 696-707, 2006.

[22] B. Kwanyuen, and D. G. Fontane, ‘‘Heuristic branch-and-bound method for ground water development planning,’’ J. Water Resour. Plann. Manage., vol. 124, no 3, pp. 140-148, 1998.

[23] S. Liberatore, G. M. Sechi, and P. Zuddas, ‘‘Non linear optimization models in water resource systems,’’ in J. Pinter, Ed, Global Optimization: Scientific and Engineering Case Studies, New York, USA: Springer, pp. 227-242, 2006.

[24] X. Cai, D. C. McKinney, L. S. Landon, and D. W. Watkins Jr, ‘‘Solving large nonconvex water resources management models using generalized Benders decomposition,’’ Oper. Res., vol. 49, no 2, pp. 235-246, 2001.

[25] Md. K. Ali, J. J. Fletcher, and T. T. Phipps, ‘‘Stream water quality management: a stochastic mixed-integer programming model,’’ Annual Meeting of the American Agricultural Economics Association (AAEA): Long Beach, CA., USA, July, 2002. Available: http;//ageconsearch.umn.edu/bitstream/19811/1/sp02al04.pdf.

[26] K. Hägglöf, ‘‘The implementation of the stochastic branch-and-bound method for applications in river basin water quality management, ’’. WP 96-89, Laxenburg, Austria: IIASA, 1996.

[27] A. A. Keller, ‘‘Multiple-use water resources management by using fuzzy multi-objective heuristic optimization methods: an overview,‘’ Int. J. Fuzzy Syst. Adv. Appl., vol. 1, no 4, pp. 36-54, 2014. Available: http://www.naun.org/cms.action?id=7628.

[28] A. A. Keller, ‘‘Convex relaxation methods for nonconvex polynomial optimization problems’’. in Mathematics and Computers in Sciences and Industry, Proc. Intern. Conf., Varna, Bulgaria, September 2014, pp. 36-45. Available: http://www.europment.org/library/2014/books/bypaper/MCSI/MCSI.pdf

[29] A. A. Keller, ‘‘Convex underestimating relaxation techniques for nonconvex polynomial programming problems: computational overview,’’ J. Mech. Behav. Mater., in press, 2015

[30] V. I. Norkin, G. Ch. Pflug, and A. Ruszczynski, ‘‘A branch and bound method for stochastic global optimization,’’ Math. Prog., vol. 83, pp. 425-450, 1998.

[31] L. Jenkins, ‘‘Parametric mixed integer programming: an application to solid waste management,’’ Manage. Sci., vol. 28, no 11, pp. 1270-1284, 1982.

[32] K. Ponnambalam, ‘‘Optimization in water reservoir systems,’’ in P.M. Pardalos and M.G.C. Resende, Eds, Handbook of Applied Optimization, Oxford, UK: Oxford University Press, pp. 933-943, 2002.

[33] R. H. Byrd, J. Nocedal, and R. A. Waltz, ‘‘KNITRO: an integrated package for nonlinear optimization,’’ in G. Di Pillo and M. Roma, Eds, Large-Scale Nonlinear Optimization, New York, USA: Springer, pp. 35-39, 2006.

[34] Ziena Optimization, ‘‘ KNITRO™ Release 9.0. Knitro Documentation,’’ Evanston IL, USA, Ziena Optimization LCC, 2013. Available: http://www.artelys.com/uploads/pdfs/Knitro90_UserManual.pdf

[35] I. Quesada and I. E. Grossmann, ‘‘An LP/NLP based branch-and-bound algorithm for convex MINLP optimization problems,’’ Comput. Chem. Eng., vol. 16, pp. 937-947, 1992.

[36] D. P. Loucks and E. Van Beek, Water Resources Systems Planning and Management: an Introduction to Methods, Models and Applications, Paris: UNESCO Publishing, 2005.

[37] Lindo Systems, LINGO User’s Guide, Chicago, IL, USA: Lindo Systems Inc, 2004.

[38] L. Schrage, Optimization Modeling with LINDO, 5th edition, Pacific Grove CA, USA: Duxbury Press, 1997.

APPENDIX A: ILLUSTRATIVE EXAMPLE OF THE GBD ALGORITHM

DOI: 10.5963/JWRHE0402003

The following illustrative example is drawn from Li & Sun [14], pp. 374-375. The MINLP problem is described by Equation (9).

$$\begin{aligned}
&\underset{x\in X, y\in Y}{\text{minimize}} \quad f(x,y) = -2\ln(1+x) + 5y \\
&\text{subject to: } 1)\ g_1(x,y) = -1 + \exp\left(\frac{x}{2}\right) - \frac{\sqrt{y}}{2} \le 0, \\
&\qquad 2)\ g_2(x,y) = 2.5 - 2\ln(1+x) - y \le 0, \\
&\qquad 3)\ g_3(x,y) = -4 + x + y \le 0, \\
&\qquad 4)\ x \in X = [0,2],\ y \in Y = \{1,3\},
\end{aligned} \tag{9}$$

where f and g_i, $i = 1,2,3$ are convex and differentiable functions; X is a compact set; and Y is a finite integer set.

The feasible region is non-connected and consists of two line segments as shown in Fig. 1. The optimal solution is $(\hat{x}, \hat{y}) = (1.0694, 2)^T$. For example, the sub-problem of iteration 1 is obtained by relaxing the integrality restriction on y. Taking $y^1 = 3$, we obtain the following continuous NLP primal $\mathcal{P}(y^1)$ sub-problem shown by Equation (10) at Step 1.

$$\begin{aligned}
&\underset{x\in[0,2]}{\text{minimize}} \quad 15 - 2\ln(1+x) \\
&\text{subject to: } 1) - 1.8660 + \exp\left(\frac{x}{2}\right) \le 0, \\
&\qquad 2) - 0.5 - 2\ln(1+x) \le 0, \\
&\qquad 3) - 1 + x \le 0.
\end{aligned} \tag{10}$$

Obtain $x^1 = 1$; the upper bound $UB^1 = 13.6137$; and deduce the duals $\lambda = (0,0,1)^T$.

At Step 2, solve the ILP problem shown in Equation (11).

$$\begin{aligned}
&\underset{y\in\{1,3\}}{\text{minimize}} \quad \alpha \\
&\text{subject to: } 15 - 2\ln(2) + 6(-3+y) \le \alpha.
\end{aligned} \tag{11}$$

Obtain $y^2 = 1$ and a lower bound $LB^1 = 1.6137$.

DOI: 10.5963/JWRHE0402003

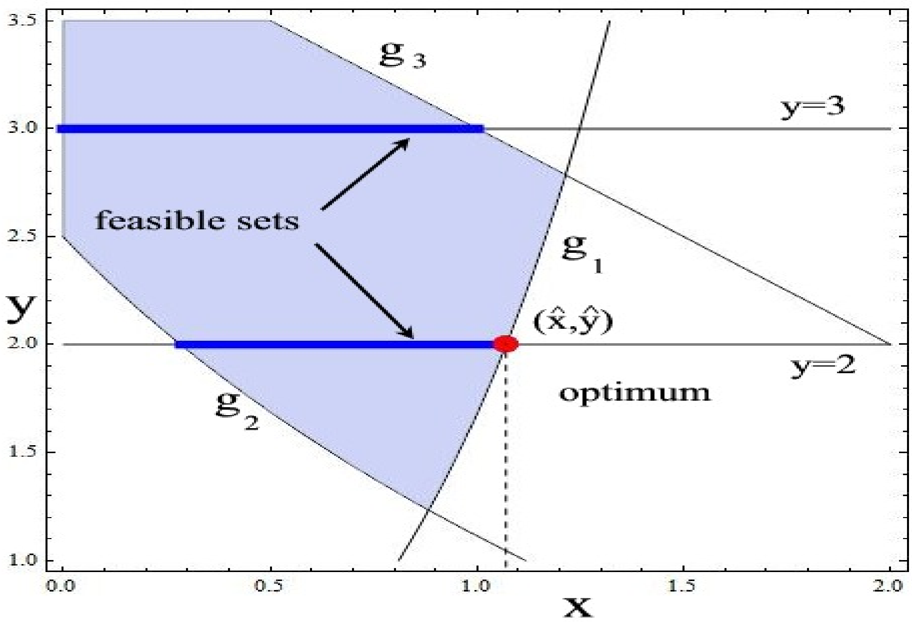

Fig. 1 Feasible sets and solution of the illustrative example

The GBD is applied to solve the MINLP problem. Three iterations are needed. All results are shown in table 3 and the finite sequence of NLP sub-problems and master ILP problems are shown in Fig. 2. NLP sub-problems provide upper bounds, whereas master ILP problem yield lower bounds to the optimal solution.

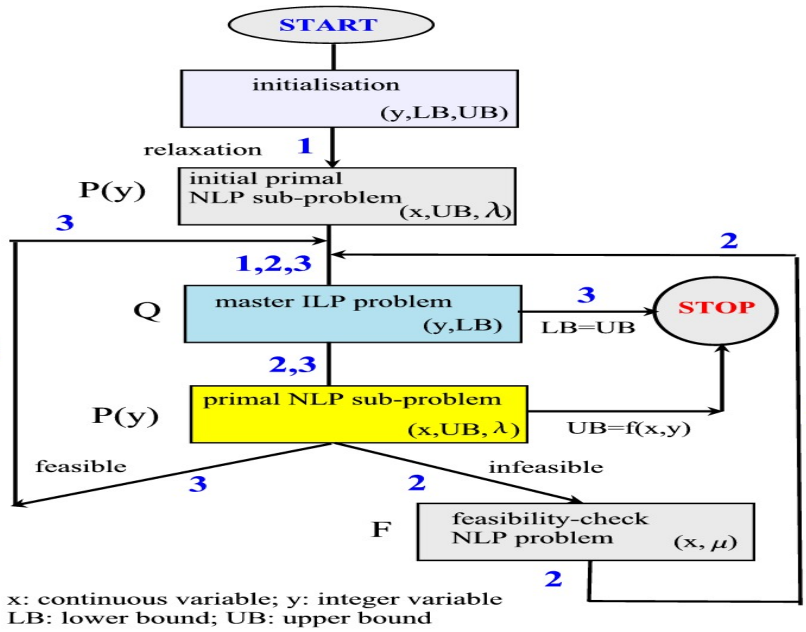

Fig. 2 Flow chart of the GBD algorithm in the illustrative example

.

DOI: 10.5963/JWRHE0402003

TABLE 3 ITERATIONS OF THE GBD ALGORITHM IN THE ILLUSTRATIVE EXAMPLE

Index		**Objective function** $^{(a)}$			**Lagrange multipliers**						**Bounds**		**Index sets**	
k	#	x^k	y^k	$f(\mathbf{x}^k)$	λ_1^k	λ_2^k	λ_3^k	μ_1^k	μ_2^k	μ_3^k	LB^k	UB^k	I^k	J^k
0	0	-	3	-	-	-	-	-	-	-	$-\infty$	$-\infty$	$\varnothing$	$\varnothing$
1	1	1	3	13.614	0	0	1	-	-	-	$-\infty$	13.6137	{1}	$\varnothing$
	2	1	1	5	-	-	-	-	-	-	1.6137	-	-	-
2	1	0.9808	-	-	-	-	-	0.554	0.447	0	1.6137	13.6137	{1}	{2}
	2	0.9808	2	8.6330	-	-	-	-	-	-	7.6137	-	-	-
3	1	1.0694	2	8.5453	1.132	0	0	-	-	-	7.6137	8.5453	{1,3}	{2}
	2	1.0694	2	8.5453	-	-	-	-	-	-	8.5453	-	-	-

$^{(a)}$ $\mathbf{x} \equiv (x, y)$, where x is a continuous variable and y an integer.

APPENDIX B: SIMPLE GROUNDWATER SUPPLY SYSTEM SERVING INDUSTRY[8]

Suppose that an industry plans to obtain water resources from a groundwater aquifer. Suppose that two well fields A and B can provide this water resource under various technical and economic conditions. The well fields have different known capacity limits. Their cost and benefit functions are piecewise nonlinear. Two questions can be submitted to the planner: how should a given water demand be distributed across the two well fields, and how should this distribution vary with the total water demand? We use a simple mixed linearized programming model to show these solutions.

The cost and sales functions are linearized over segments of demand values with possible shifts (i.e., discontinuities). Binary 0-1 parameters are introduced to specify piecewise-linear functions and capacities, as shown in Fig. 3(a). The mixed linear programming problem is given in Equation (12).

$$
\begin{aligned}
&\underset{\mathbf{x}\in\mathbf{R}_+^{10},\,\mathbf{y}\in\{0,1\}^5}{\text{minimize}} \quad x_1 + x_2 \\
&\text{subject to:}\ 1)\ x_1 = (8y_1 + 2.4x_4) + (26y_2 + 0.8x_5) + (35y_3 + 0.7143x_6), \\
&\qquad 2)\ x_2 = (15y_4 + x_8) + (18y_5 + 0.2857x_9) + 2x_{10}, \\
&\qquad 3)\ x_3 = x_4 + (5y_2 + x_5) + (10y_3 + x_6), \\
&\qquad 4)\ x_4 \le 5y_1,\ x_5 \le 5y_2,\ x_6 \le 7y_3, \\
&\qquad 5)\ y_1 + y_2 + y_3 \le 1, \\
&\qquad 6)\ x_7 = x_8 + (3y_5 + x_9) + x_{10}, \\
&\qquad 7)\ x_8 \le 3y_4,\ x_9 \le 7y_5,\ x_{10} \le 3y_5, \\
&\qquad 8)\ y_4 + y_5 \le 1, \\
&\qquad 9)\ x_3 + x_7 \le Q,
\end{aligned}
\tag{12}
$$

where decision variables x_1 and x_2 are continuous total costs (i.e., fixed and variable costs) for well fields A and B; x_3 represents the continuous total flow of A and x_4, x_5, x_6 are the continuous flows of A on segments of demand values Q ; x_7

[8] This illustrative model is adapted from numerical examples by Loucks & Van Beek (2005)[36]. Results are same with different notations and illustrations. This version has been solved by using the LINGO package [37] and LINDO 6.1 (see Schrage (1997)[38]).

DOI: 10.5963/JWRHE0402003

is the continuous total flow of B and x_8, x_9, x_{10} are the continuous flows of B on segments of demand values; and y_1 to y_5 are binary variables. The maximum flow capacity of well fields A and B are set to 17 and 13, respectively.[9]

The model is solved for various values of water demand from the industry. Fig. 3 (b) and Fig. 4 show the optimal least-cost solutions when the total demand Q varies. In Fig. 4, well field A supplies all water demand for $Q \le 4.3$ and $15 \le Q \le 17$; well field B provides all water demand for $4.3 \le Q \le 13$. For $13 \le Q \le 15$ and $17 \le Q \le 18$. Well field B supplies its full flow capacity and well field A provides a required additional amount of water. Suppose that the piecewise-linear approximated function of sales is represented by:

$$S = \begin{cases} -10 + 3.8889Q, \text{ for } Q \in [0,9] \\ 10 + 1.6667Q, \text{ for } Q \in [9,21] \\ 33 + 0.5556Q, \text{ for } Q \in [21,30] \end{cases} \tag{13}$$

Profit is obtained if total sales S exceed the total costs $C = x_1 + x_2$ as shown in Fig. 3 (b), in which water demands range in the interval $[8.1580, 14.2269]$.

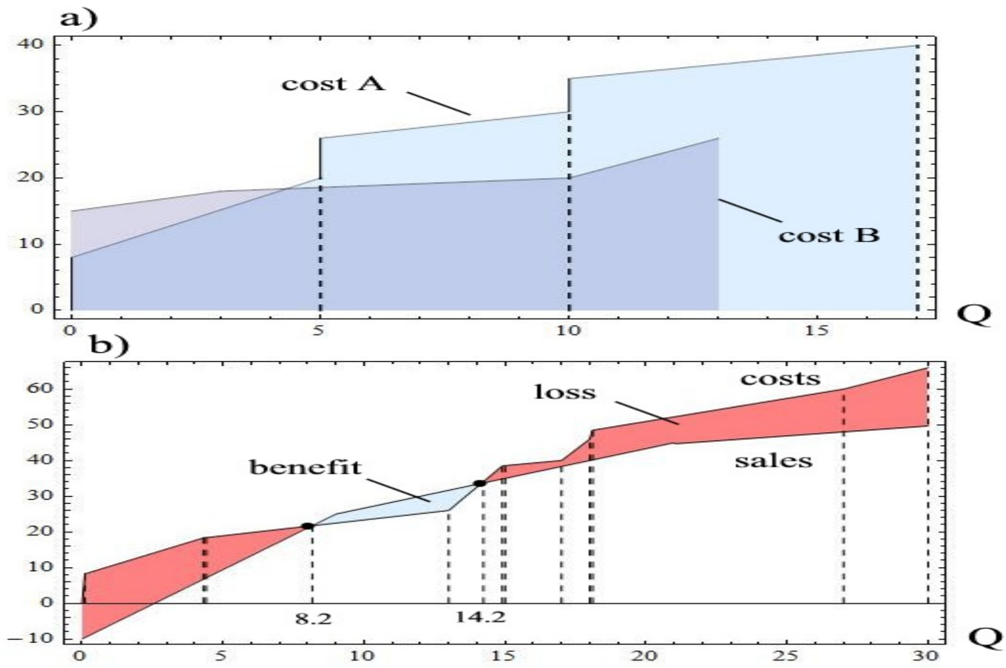

Fig. 3(a) Cost functions of well fields A and B; b) total optimal sales, costs, losses and benefits for various water demands from the industry

[9] In fact, combining the constraints of well field A for x_3 to x_6, we obtain $x_3 \le 5y_1 + 10y_2 + 17y_3$. Then we get $x_3 \le 17$ since $y_1 = y_2 = 0$, and $y_3 = 1$.

DOI: 10.5963/JWRHE0402003

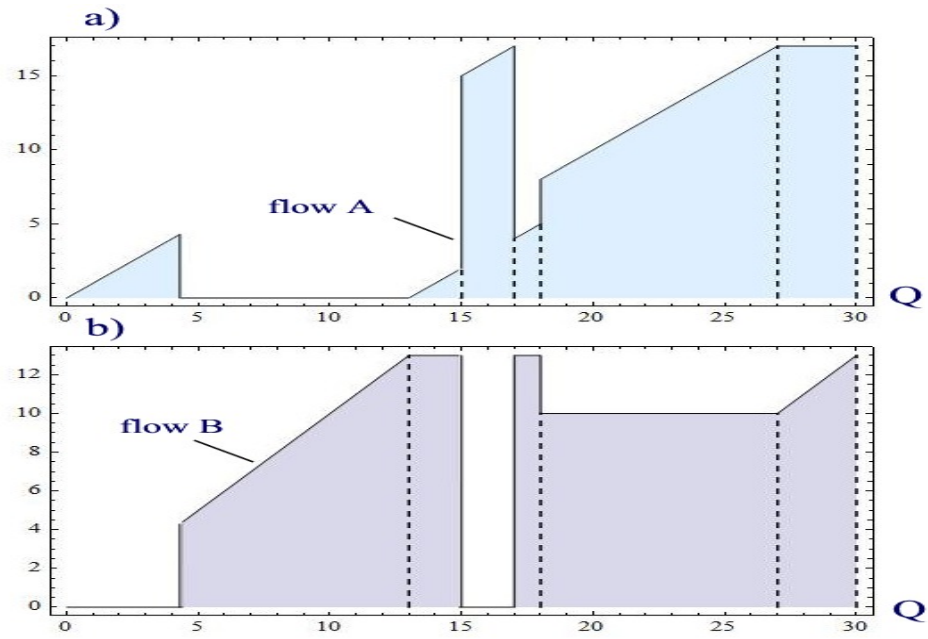

Fig. 4(a) Optimal flow of well field A for different values of total demand; b) optimal flow of well field A for different values of water demands from the industry

DOI: 10.5963/JWRHE0402003