Tarzan's Dilemma for Elliptic and Cycloidal Motion

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***Abstract*-In this paper, we study Tarzan's dilemma of elliptic and cycloidal motion. We give the relation between the flying distance and the launching angle of these two different motions, and perform numerical calculation to find the launching angle which maximizes the flying distance. This study will be helpful for understanding elementary mechanics for students.**

Keywords- Classical Mechanics; Physics Education

I. INTRODUCTION

When one throws a ball from the ground into the air with definite speed and launching angle θ, what is the launching angle to maximize the flying distance? The answer 45^o of this well-known problem can be derived from simple calculations of Newtonian mechanics. Next, let us consider a situation that an object (saying "Tarzan") is attached at one end of rope and the other end is fixed at a point above the ground. If Tarzan starts swinging due to the gravity and releases his hands from the rope at an intermediate point, he is thrown into the air with corresponding launching angle θ_s. The problem is that what angle θ_s gives the maximal flying distance. If Tarzan wants to get large θ_s (closed to 45^o), the launching speed is slow because he loses kinetic energy. If he wants to get large launching speed, he cannot get enough large launching angle. Therefore this problem is called "Tarzan's dilemma" [1–5].

In previous works [1–5], the authors have considered the case of ordinary circular motion. However, this is not the only way to jump when Tarzan has a rope of definite length. If both ends of rope are fixed at each point above the ground, and Tarzan holds midpoint of the rope, his trajectory of the swing will be an ellipse, horizontally or vertically long ellipse. If one end of the rope is fixed at a point between two inverted cycloidal objects, Tarzan's trajectory will be a cycloidal curve. In any case, one can find the launching angle θ_s which maximizes the flying distance L and compare that what way of swing will be the best for Tarzan. In this paper, we consider Tarzan's dilemma in elliptic and cycloidal motion and perform numerical calculation to find the best launching angle θ_s, comparing to those of ordinary circular motion. In educational point of view, solving this problem develops student's qualitative and conceptual understanding for mechanical motion and energy conservation law without tough calculation. For example, what is the best way of swing to reach the farthest distance and what is the magnitude relation of θ_s between three ways?

In the paper, we discuss the case of elliptic motion and cycloidal motion and derive a formula for the flying distance L in analogy with the case of circular motion. Besides, we discuss the motion of Tarzan in detail by using the results of previous sections. Comparing four ways of swinging, we find that the flying distance is the largest in the case of circular motion when the length of rope is definite.

II. ELLIPTIC TARZAN

We discuss Tarzan's jump in the case of elliptic motion. As shown in the left panel of Figure 1, let us consider the situation that Tarzan has a rope of the length $2a$, and fixes it at two points F and F' with the distance $2\sqrt{a^2-b^2}$, with a > b. He stands at the starting point A holding the midpoint of the rope and starts to swing. If friction between the rope and his hands is negligible, he moves along an elliptic curve with semi-major (semi-minor) axis $a(b)$ due to the gravity and tension from the rope. After passing the lowest point B of the height h, he flies into the air by releasing the rope at point C, and finally lands on point D.

Now we reinterpret this situation as swing with an imaginary rope fixed at one point, as shown in the right panel of Figure 1. This imaginary rope changes its length such that Tarzan can move along elliptic curve. While it is difficult to discuss tension of such rope, one does not have to worry about it because only geometrical consideration is needed for calculation. Although we discuss the case of a > b with eccentricity $e=\sqrt{(a^2-b^2)/a^2}$, the following discussions are valid for the case of a < b as well by using $e'=\sqrt{(b^2-a^2)/b^2}$. In the elliptic case, the length of the elastic rope varies as a function of position unlike the ordinary circular case. Tarzan starts swinging from the starting point A with the position

$$x_A=-r_\alpha \sin\alpha, y_A=h+b-r_\alpha\cos\alpha \tag{2.1}$$

where

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$$r_\alpha = a\sqrt{\frac{1-e^2}{1-e^2\sin^2\alpha}}\,. \tag{2.2}$$

For $t_s \le t \le t_f$, the position of Tarzan in the air is given by

$$x(t) = r_s \sin\phi_s + v_s t\cos\theta_s\,, \tag{2.3}$$

$$y(t) = h + b - r_s\cos\phi_s + v_s t\sin\theta_s - \frac{1}{2}gt^2\,, \tag{2.4}$$

where r_s is defined similar to Eq. (2.2) by replacing α to the angle of the launching point C, ϕ_s. g is gravitational acceleration. The angle ϕ_s and the launching angle θ_s is related as

$$\tan\theta_s = \frac{b^2}{a^2}\tan\phi_s\,. \tag{2.5}$$

At $t = t_f$, Tarzan lands on the ground of point D with the position

$$L = r_s\sin\phi_s + v_s t_f\cos\theta_s\,, \tag{2.6}$$

$$0 = h + b - r_s\cos\phi_s + v_s t_f\sin\theta_s - \frac{1}{2}gt_f^2\,. \tag{2.7}$$

From Eqs. (2.6), (2.7) and the energy conservation law between point A and point C

$$mg[h + b - r_\alpha\cos\alpha] = \frac{1}{2}mv_s^2 + mg[h + b - r_s\cos\phi_s], \tag{2.8}$$

we obtain the flying distance L from the origin as a function of θ_s

$$L = r_s\sin\phi_s + a\Delta_{EL}\sin 2\theta_s + \sqrt{(a\Delta_{EL}\sin 2\theta_s)^2 + 4a\Delta_{EL}\cos^2\theta_s[h + b - r_s\cos\phi_s]}, \tag{2.9}$$

where

$$\Delta_{EL} = \frac{1}{a}(r_s\cos\phi_s - r_\alpha\cos\alpha)\,, \tag{2.10}$$

corresponds to the difference in height between point A and point C. One can check that in the case of $a = b(= r)$, Eq. (2.9) reduces to that of the circular case [2]. The problem is to find the value of θ_s which maximizes L for given a, b, h and α. The above discussions are valid for the case of $a < b$ by replacing $r_{\alpha,s}$ to $r'_{\alpha,s}$ defined as

$$r'_\alpha = a\sqrt{\frac{1}{1-e'^2\cos^2\alpha}},\; r'_s = a\sqrt{\frac{1}{1-e'^2\cos^2\phi_s}}\,. \tag{2.11}$$

We perform numerical calculation in section 5 for both cases $a > b$ (the case EL-1) and $a < b$ (the case EL-2).

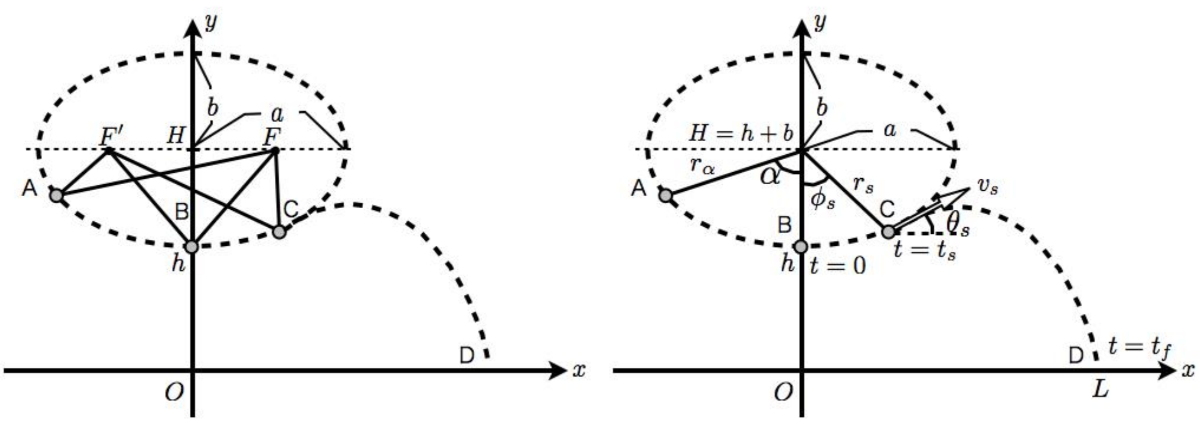

Figure 1 Tarzan's jump of elliptic motion. Left: Tarzan swings with the rope of length 2a which is fixed at two points F and F'. Right: Tarzan's jump with an elastic rope fixed at one point. We perform calculation and analyze the motion in the notation of the right panel for convenience.

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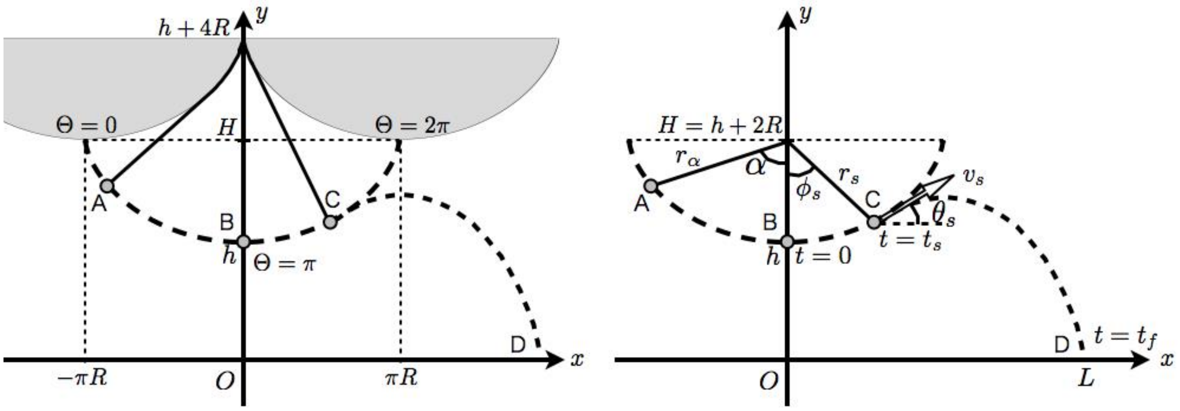

Figure 2 Tarzan's jump of cycloidal motion. Left: Tarzan swings with the rope of length 4R, which is fixed at a point between two cycloidal objects. Right: Equivalent figure of Tarzan's jump with an imaginary elastic rope whose length can be changed such that Tarzan moves along the cycloidal curve.

III. CYCLOIDAL TARZAN

As the next example of Tarzan's swing, we discuss the case of cycloidal swing of Tarzan in this section. As shown in the left panel of Figure 2, if a rope of the length 4*R* is fixed at a point of the height *h* + 4*R* between two inverted cycloidal objects generated from a circle with radius *R*, Tarzan moves along cycloidal curve while swinging. The parametric form of the resulting cycloid of Tarzan is given by

$$x = R(\Theta - \pi - \sin\Theta),\ y = h + R(1+\cos\Theta)\,, \tag{3.1}$$

with $0 \le \Theta \le 2\pi$. Again we reinterpret this cycloidal motion as swing with an imaginary elastic rope fixed at a point of the height $H = h+2R$ as depicted in the right panel of Figure 2. The length of the rope can be changed from 2*R* to πR such that its end of the rope (Tarzan) moves along the cycloidal curve. We define angles α for $0 \le \Theta < \pi$ and ϕ for $\pi \le \Theta \le 2\pi$ measuring from the *y*-axis, as shown in the Figure 2. The angles Θ_α and Θ_s corresponding to point A and point C are related to the angle α and ϕ_s as given below.

Tarzan starts swinging from point A with the position

$$x_A = -r_\alpha \sin\alpha,\ y_A = h + 2R - r_\alpha \cos\alpha\,, \tag{3.2}$$

where

$$\tan\alpha = -\frac{\Theta_\alpha - \pi - \sin\Theta_\alpha}{1-\cos\Theta_\alpha},\ r_\alpha = R\sqrt{(\Theta_\alpha - \pi - \sin\Theta_\alpha)^2 + (1-\cos\Theta_\alpha)^2}\,. \tag{3.3}$$

After launching from point C with the angle θ_s, at $t = t_s$, Tarzan in the air, for $t_s \le t \le t_f$, is at the position

$$x(t) = r_s \sin\phi_s + v_s t\cos\theta_s\,, \tag{3.4}$$

$$y(t) = h + 2R - r_s\cos\phi_s + v_s t\sin\theta_s - \frac{1}{2}gt^2\,, \tag{3.5}$$

where

$$\tan\phi_s = -\frac{\Theta_s - \pi - \sin\Theta_s}{1-\cos\Theta_s},\ r_s = R\sqrt{(\Theta_s - \pi - \sin\Theta_s)^2 + (1-\cos\Theta_s)^2}\,, \tag{3.6}$$

and

$$\tan\theta_s = -\cot\frac{\Theta_s}{2} \tag{3.7}$$

At $t = t_f$, Tarzan lands on the ground at point D with the position

$$L = r_s\sin\phi_s + v_s t_f\cos\theta_s\,, \tag{3.8}$$

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$$0 = h + 2R - r_s \cos\phi_s + v_s t_f \sin\theta_s - \frac{1}{2} g t_f^2 \cdot \tag{3.9}$$

The energy conservation law between point A and point C is given by

$$mg[h + 2R - r_\alpha \cos\alpha] = \frac{1}{2} m v_s^2 + mg[h + 2R - r_s \cos\phi_s], \tag{3.10}$$

as usual. From Eqs. (3.8), (3.9) and (3.10), we obtain the flying distance L from the origin as

$$L = r_s \sin\phi_s + R\Delta_{CY} \sin 2\theta_s + \sqrt{(R\Delta_{CY} \sin 2\theta_s)^2 + 4R\Delta_{CY} \cos^2\theta_s [h + 2R - r_s \cos\phi_s]}, \tag{3.11}$$

where

$$\Delta_{CY} = \frac{1}{R}(r_s \cos\phi_s - r_\alpha \cos\alpha), \tag{3.12}$$

has been defined as the case of the previous section. In the next section, we perform numerical calculation to find the value of θ_s which maximizes L. We call the motion of this type the case CY.

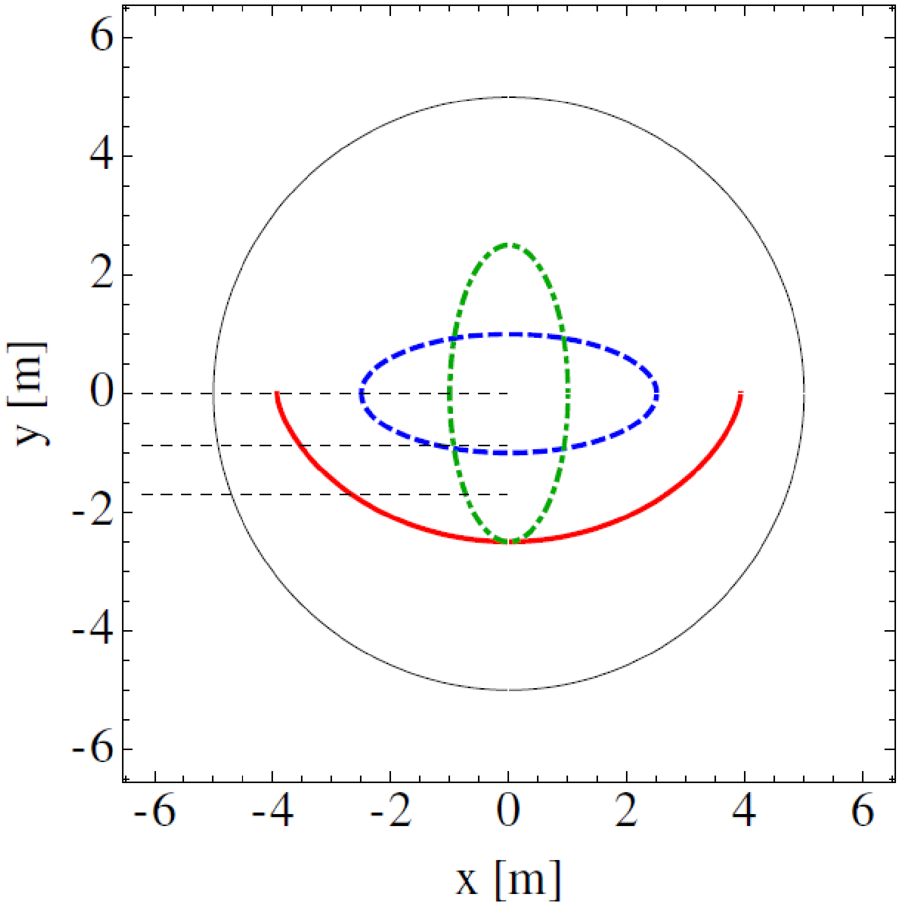

Figure 3 Tarzan's trajectory for CI (black solid), EL-1 (blue-dashed), EL-2 (green dot-dashed), and CY (red thick). Three horizontal dashed lines correspond to H1(2), M1(2) and L1(2) from top to bottom.

IV. DISCUSSIONS

In this section, we discuss relation between the flying distance L and the launching angle θ_s in the cases given in the previous sections, such as circular (CI) [2], elliptic (EL-1 and EL-2) and cycloidal (CY) Tarzan. We perform numerical calculation and compare differences between swings under the following conditions. (1) The length of rope is 5.0 m. We set r = 5.0 m (CI), (a, b) = (2.5, 1.0) m (EL-1), (a, b) = (1.0, 2.5) m (EL-2), and R = 1.25 m (CY). (2) When the starting angle α = 90 [deg.], the starting point A is H = 20 (or 5.0) m high. For comparison, we set the height of point A to be (20, 19.1, 18.3) m referring to case (H1, M1, L1), and (5.0, 4.1, 3.3) m referring to case (H2, M2, L2). Relative height and trajectory are shown in Figure 3. For the case L1 and L2 in EL-2, there is no solution because the starting point A is below the lowest point B.

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Figure 4 show the relation between the flying distance L and the launching angle θ_s for H = 20 m (upper three panels) and H = 5.0 m (lower three panels). In each panel, black solid, blue dashed, green dot-dashed and red thick curve corresponds to CI, EL-1, EL-2 and CY, respectively. Since the height h of the lowest point B is the same for EL-2 and CY, green dot-dashed curve and red thick curve crosses at θ_s = 0 in any case. One can see that the CI case gives the longest distance under any condition because Tarzan can get the largest kinetic energy while swinging. For H = 5.0 m cases (H2, M2 and L2), the flying distance L of CI case is smaller than that of other cases for relatively small θ_s unlike H = 20 m cases (H1, M1 and L1), because Tarzan touches the ground at the lowest point B (θ_s = 0) and therefore he cannot get large L unless θ_s becomes large.

Obviously each curve has the maximal value. The maximal value of the flying distance L and the corresponding launching angle θ_s are given in Table 1 for all cases of Figure 4. In each element of Table 1, the values (α [deg.], θ_s [deg.], L[m]) are given, where L is the maximal value for given height of point A and θ_s is the corresponding launching angle. As shown in Table 1, the launching angle θ_s to get the longest L becomes smaller when the starting point A becomes lower. The reason is that as the point A becomes lower, larger horizontal speed $v_s \cos\theta_s$ is needed to get longer flying distance L and the flying time $t_f - t_s$ becomes shorter. Comparing the four cases CI, EL-1, EL-2 and CY, as the lowest point B is lower, the flying distance L becomes larger because Tarzan can get larger kinetic energy. However for EL-2 and CY with the same height of point B, the maximal value of L for CY is always larger than that of EL-2 because the x-coordinate $x(t_s)$ for CY is larger than that of EL-2, depending on curvature.

As for the launching angle θ_s, the magnitude relation between four ways is

$$\theta_s(CI) > \theta_s(EL-2) \geq \theta_s(CY) > \theta_s(EL-1) \tag{4.1}$$

in any case. This relation depends on what kinetic energy Tarzan can get. This can be seen in Figure 5. Figure 5 is the trajectory of Tarzan for the case of α = 90 [deg.]. The trajectories of EL-1 and CY are similar because the lowest point B is the same in height in both cases. The maximal value of the flying distance *L* strongly depends on the height *h* of the lowest point B. However, it depends on θ_s and type of the swing as well. The general discussion of dependence of *L* on *h* is beyond the scope of this manuscript.

In the educational point of view, this problem requires conceptual understandings of physics for students as well as some calculations. Students can derive the relation between *L* and θ_s without tough calculation, only by laws of projectile moving due to the gravity and energy conservation law. Although the θ_s dependence of *L* seems complicated, students can find that there exists the maximal value of *L* and discuss the reason of the relation Eq. (4.1) given above. Considering and discussing such physical meanings will be helpful for students to understand physics.

Table 1 The values (α [deg.], θ_s [deg.], L [m]) for definite height of point A. L is the maximal value for given height of point A, and θ_s is the corresponding launching angle.

Height of A[m]	CI	EL-1	EL-2	CY
20(H1)	(90, 25.4, 21.2)	(90, 11.9, 9.58)	(90, 19.4, 14.2)	(90, 18.5, 15.0)
19.1(M1)	(80, 23.8, 18.9)	(55.0, 4.41, 3.42)	(47.2, 16.2, 11.2)	(76.2, 15.4, 11.9)
18.3(L1)	(70, 21.8, 16.5)	No	(23.1, 11.7, 7.64)	(57.2, 11.1, 8.10)
5.0(H1)	(90,41.0, 12.0)	(90, 20.8, 5.53)	(90, 34.8, 7.21)	(90, 31.2, 8.52)
4.1(M1)	(80, 40.2, 10.2)	(55.0, 7.81, 1.94)	(47.2, 31.6, 5.32)	(76.2, 27.3, 6.49)
3.3(L1)	(70, 39.1, 8.50)	No	(23.1, 25.5, 3.32)	(57.2, 20.8, 4.24)

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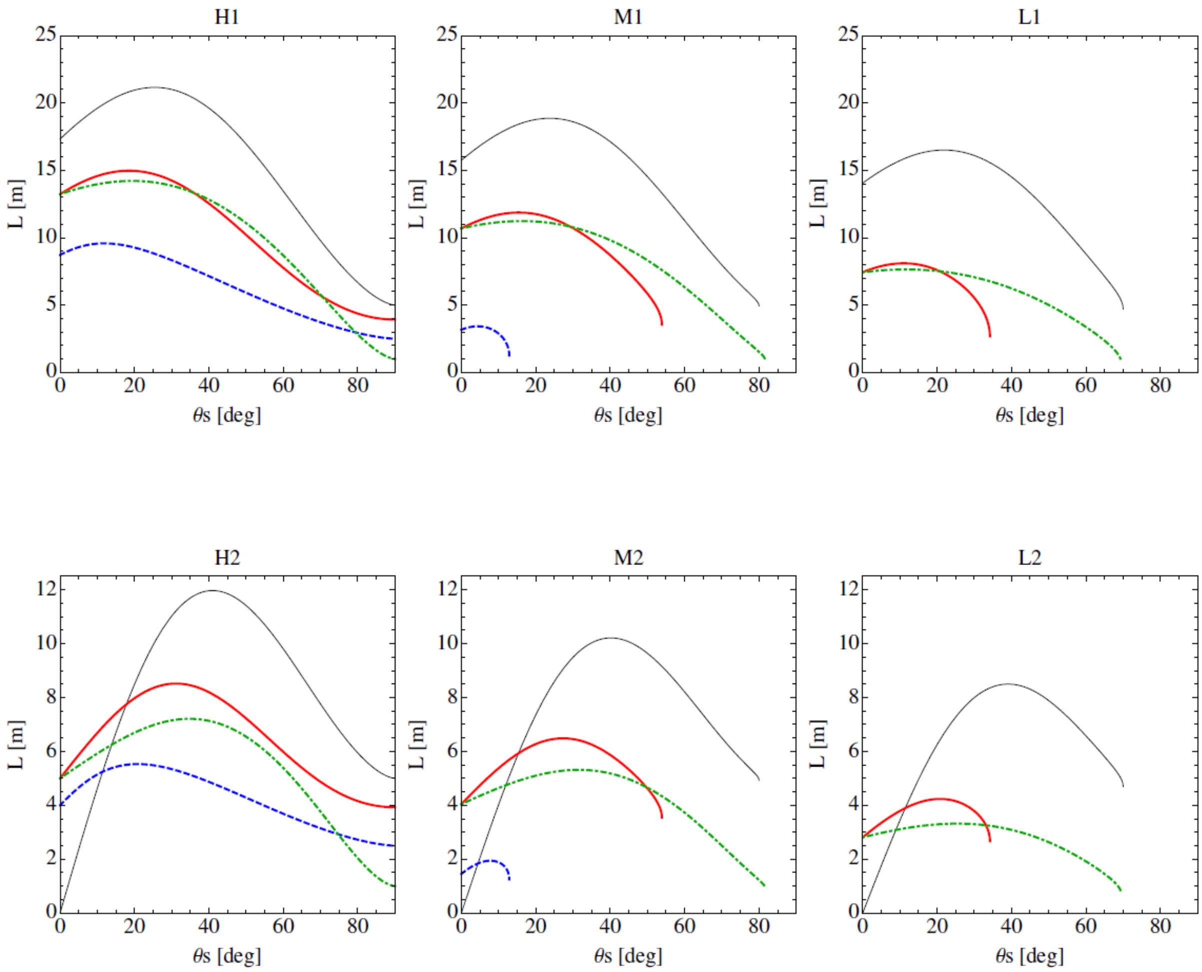

Figure 4 Flying distance L as a function of the launching angle θ_s [deg.] for H = 20 m (upper three panels) and H = 5.0 m (lower three panels). In each panel, black solid, blue dashed, green dot-dashed and red thick curve corresponds to CI, EL-1, EL-2 and CY, respectively.

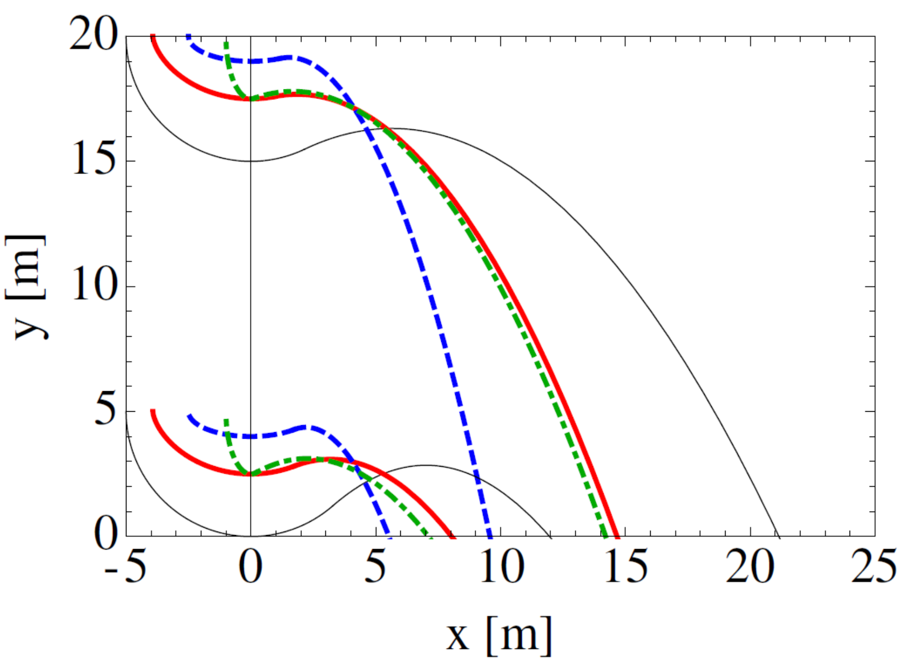

Figure 5 Tarzan's trajectory in the case of α = 90 [deg.] for CI (black), EL-1(blue dashed), EL-2 (green dot-dashed), CY (red thick) motion, for H = 20 m (upper four curves) and H = 5.0 m (lower four curves). The length of the rope is 5.0 m in all cases.

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V. CONCLUSIONS

We have studied Tarzan's dilemma in three different motions, including circular, elliptic and cycloidal swing. These three motions can be realized by fixing rope in different ways. We have derived formulae of the flying distance L as a function of the launching angle θ_s by simple calculation and performed numerical calculations to find the maximal value of L and corresponding θ_s for each type of swing. There exists the value of θ_s which maximizes L, and its value of θ_s increases as the height of the lowest point B decreases, depending on the type of swing. As the lowest point B becomes lower, Tarzan can get larger kinetic energy and therefore θ_s becomes larger (close to 45^o). In our calculation, we have set that the case CY and EL-2 have the same lowest point in height. In such a case, trajectory of Tarzan is similar with each other, and the difference comes from curvature of swing. Calculation and physical interpretation to solve this Tarzan's dilemma will be helpful for students to develop conceptual understanding of physics as well as quantitative calculation.

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