

Adapted MacCormack Finite-Differences Scheme for Water Hammer Simulation

Lyes Amara¹, Ali Berreksi², Bachir Achour³

^{1,3}Research Laboratory in Subterranean and Surface Hydraulics (LARHYSS), University of Biskra, P.O. Box 145, R.P., 07000, Biskra, Algeria

²University of Bejaia, Hydraulic Department, Targa Ouzemour, 06000, Bejaia, Algeria, LRS-Eau Laboratory, Polytechnic National School of Algiers, Algeria

¹amara.lyes@yahoo.fr; ²ali_berreksi@yahoo.fr; ³bachir.achour@larhyss.net

Abstract-An adapted second-order accurate MacCormack finite-differences scheme is introduced and tested for the integration of the water hammer equations for a friction pipe. A fractional method is used to solve the governing equations in two steps with the Runge-Kutta splitting technique. The details of the proposed improvement technique, boundary condition inclusion and the shock capturing capability are presented in this paper. The numerical oscillations resulting from the dispersive errors of the MacCormack original scheme are treated using the artificial viscosity procedure. The results computed using the adapted MacCormack scheme for a friction pipeline with the original scheme with numerical viscosity are compared and analyzed. It is shown that for an abrupt varied flow, the proposed technique leads to better results.

Keywords- Water Hammer; MacCormack; Numerical Oscillations; Artificial Viscosity; Splitting Technique; Runge-Kutta

I. INTRODUCTION

The study of hydraulic transients or water hammer problems has a great significance in wide range of industrial and municipal applications, such as power plants, water supply systems, sewage pipelines...etc. Thus accurate modeling of water hammer events is vital for proper design and safe operation of pressurized pipes. Designs of pipeline systems, and the prediction of the system behavior, hence require efficient mathematical models capable of accurately solving water hammer problems. However, the mathematical model governing the phenomenon is a set of quasi-linear, hyperbolic, partial differential equations [1]. Because of the presence of the nonlinear source term representing the losses due to the friction, these equations can only be numerically integrated. Due to the abrupt flow changes, steep waves or “shocks” may be generated at different boundaries, and any numerical method used should be able to handle these shocks.

Among various methods, the method of characteristics (MOC), which transforms the partial differential equations into a set of ordinary differential equations, is the most popular since it provides the desirable accuracy, numerical efficiency, programming simplicity and ability to handle boundary conditions [2]. For large friction losses, Wylie and Streeter [3] show that the solution may be unstable even Courant number $C_n \leq 1$. Additionally, necessary interpolations if $C_n \neq 1$ in certain situations smear the shocks and degrade the solution. To avoid this, Chaudhry and Hussaini [4] applied three explicit second-order accurate finite-differences schemes, MacCormack, Lambda and Gabutti scheme to the water hammer equations. It was shown by the L_1 and L_2 norms for a frictionless pipe that the MacCormack scheme produces better results than the first-order MOC and fewer mesh points may be used for this scheme to achieve the same accuracy.

In this paper, the second-order MacCormack finite-difference scheme is applied and adapted for the water hammer equations with source term, i.e. for a friction pipe, in order to avoid instabilities and improve the numerical solution. First, the governing equations of water hammer are given, and then the discretized model by the MacCormack scheme is presented. Secondly, the artificial viscosity technique and the proposed Runge-Kutta splitting method for the term source treatment in the MacCormack scheme are exposed. Finally, the results computed for sudden closure valve on friction pipe are compared and discussed

II. GOVERNING DIFFERENTIAL EQUATIONS

The governing equations for water hammer flows, with unidirectional assumption, are classically derived by applying the principles of the conservation of mass and momentum to a control volume. These equations describe the axial and temporal variation of the cross-sectional average of the field variables in transient pipe flows [5]. For a pipe with constant diameter, the water hammer equations can be written as follow [1, 3]:

$$\frac{\partial H}{\partial t} + \frac{a^2}{gS} \frac{\partial Q}{\partial x} = 0 \quad (1)$$

$$\frac{\partial Q}{\partial t} + gS \frac{\partial H}{\partial x} + RQ|Q| = 0 \quad (2)$$

In which t = time, x = distance, S = cross-sectional area of the conduit, $H(x,t)$ = piezometric head above the datum, $Q(x,t)$ = discharge, $R = \lambda/(2DS)$, λ = friction factor, D = diameter of the pipe and a = wave speed. Note that the nonlinear convective terms are neglected in Eqs. (1) and (2) since they are small for the majority of the water hammer problems [2, 5]. These equations can be written in their vector divergent form as follow:

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + G = 0 \quad (3)$$

Where the vector variable U , the flux vector F and the source term vector are given by the following expressions:

$$U = \begin{bmatrix} H \\ Q \end{bmatrix} \quad F = \begin{bmatrix} \frac{a^2}{gS} Q \\ gSH \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ RQ|Q| \end{bmatrix} \quad (4)$$

Hence, the mathematical model governing the unsteady flow in pipelines is a set of quasilinear, hyperbolic, partial differential equations having constants coefficients [1]. Because of the non-linear source term G , a closed-form solution is not possible. Therefore, numerical methods such as finite-differences schemes have been used to integrate these equations.

III. THE MACCORMACK NUMERICAL SCHEME

Originally conceived for the solution of the compressible Navier-Stokes equations, the finite-differences scheme introduced by MacCormack [6] was applied first by Chaudhry and Hussaini [4] for the water hammer analysis and for the unsteady free surface flow by Fennema and Chaudhry [7] and Garcia and Kahawita [8]. The MacCormack scheme is second-order accurate in both space and time. It is comprised of two parts predictor and corrector, in which one-sided finite-difference approximations are used at each step [9].

A. General Formulation

Two alternatives of this scheme are possible. In one alternative, forward finite-differences are used to approximate the spatial partial derivatives in the predictor part and backward finite-differences are utilized in the corrector part. The values of variables determined during the predictor part are used during the corrector part [10]. The finite-difference approximations for the first alternative of this scheme are used here; equations for the second alternative of the scheme may be written similarly by reversing the direction of the spatial finite-difference approximations. Referring to Fig. 1, it can be written for the first alternative:

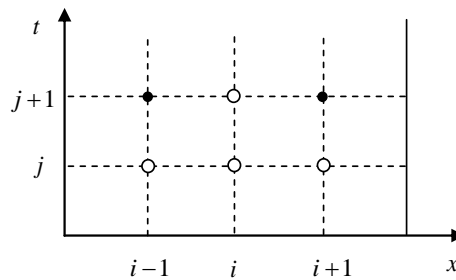


Fig. 1 Computational grid

1) Predictor Part:

$$\tilde{U}_i = U_i^j - \frac{\Delta t}{\Delta x} (F_{i+1}^j - F_i^j) - G_i^j \Delta t \quad 2 \leq i \leq N \quad (5)$$

2) Corrector Part:

$$\hat{U}_i = U_i^j - \frac{\Delta t}{\Delta x} (\tilde{F}_i^j - \tilde{F}_{i-1}^j) - \tilde{G}_i^j \Delta t \quad 1 \leq i \leq N-1 \quad (6)$$

In which \tilde{U} and \hat{U} are the intermediate values for the vector U . The new value of U at a level time $j+1$ is then obtained from:

$$U_i^{j+1} = \frac{1}{2}(\tilde{U}_i + \hat{U}_i) \quad (7)$$

IV. BOUNDARY CONDITIONS

Inclusion of boundaries is an important aspect of the numerical models for hyperbolic systems since errors introduced at a boundary will be propagated throughout the computational domain and may lead to instabilities [11].

The formulations described above solve for the values of H and Q at the interior points. At the boundaries, however, we can not use these equations since there is no grid point outside the computational domain. According to Chaudhry and Hussaini [4], boundary conditions may be included in the analysis by using the characteristic equations or by extrapolating fluxes technique. In the first procedure, which is used in this paper, we solve the positive characteristic equation simultaneously with the condition imposed by the boundary for the downstream-end condition (Fig. 2). This condition, for a valve, is a relationship between head and discharge flow or sometimes a time function of the flow variation.

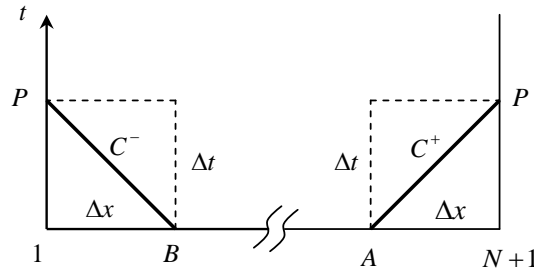


Fig. 2 Characteristic lines at boundaries

At the upstream-end, however, the boundary condition is solved simultaneously with the negative characteristic equation. Having a constant level reservoir at the upstream of the domain, the condition is that the water level remains invariable with time.

V. STABILITY

For hyperbolic equations, the zone of dependence of a dependent variable is enclosed by the characteristics curves. These characteristics directions are the lines along which disturbances are propagated in the $x-t$ plane [12]. A stable explicit finite-differences method has a numerical zone of dependence which includes the region defined by the characteristics curves. This is known as the Courant-Friedrichs-Lewy (CFL) condition. For a fixed spatial grid, the time increment must be imposed as follow:

$$\Delta t = C_r \frac{\Delta x}{a} \quad (8)$$

A von Neumann stability analysis shows that for the MacCormack scheme, the Courant number C_r must be less than or equal to 1 [7, 13].

VI. ARTIFICIAL VISCOSITY

MacCormack scheme is second-order accurate in space and time and results in dispersive errors. These dispersive errors cause high-frequency oscillations near steep gradient. In the model presented herein, a procedure developed by Jameson and al. [7] is used to dampen these numerical instabilities. This procedure, smoothes regions of large gradients while leaving smooth areas relatively undisturbed. The values of the variables at the new time computed by MacCormack method are modified using the following algorithm [10]:

$$\xi_i = \frac{|H_{i+1} - 2H_i + H_{i-1}|}{|H_{i+1}| + 2|H_i| + |H_{i-1}|} \quad (9)$$

$$\chi_{i+1/2} = \Phi \max(\xi_{i+1}, \xi_i) \quad (10)$$

In which Φ is dissipation constant to regulate the amount of artificial viscosity. At nodes near to the boundaries a one-side finite-difference approximation is used in Eq. (9). We use the following expressions instead of Eq. (9):

$$\xi_i = \frac{|H_i + H_{i-1}|}{|H_i| + |H_{i-1}|} \quad (11)$$

$$\xi_i = \frac{|H_{i+1} - H_i|}{|H_{i+1}| + |H_i|} \quad (12)$$

The computed dependent variable f is then modified as follow:

$$f_i^{j+1} = f_i^{j+1} + \chi_{i+1/2} (f_{i+1}^{j+1} - f_i^{j+1}) - \chi_{i-1/2} (f_i^{j+1} - f_{i-1}^{j+1}) \quad (13)$$

f refers here to both H and Q . The above procedure is equivalent to adding second-order dissipative terms to the original governing equations.

VII. FRACTIONAL STEP METHOD (SPLITTING TECHNIQUE)

In order to advance the solution from j to $j+1$, Eq. (3) needs to be integrated with respect to time. Instead of solving the non-homogeneous Eq. (3) directly, the proposed approach in this paper is to split the balance law (3) into two sub-problems. This idea, borrowed from the computational fluid dynamics [14], is introduced here for the water hammer simulation in order to improve the numerical solution and the source term treatment in the MacCormack finite-differences scheme. The problem is hence solved in two steps:

A. First Step (Pure Propagation)

The homogeneous part of the hyperbolic balance law (3) is first solved, i.e. without source term, to compute U_i^{j+1} using the MacCormack scheme:

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0 \quad (14)$$

B. Second Step (Damping)

The source term is introduced then in the solution as an ordinary differential equation as follow:

$$\frac{\partial U}{\partial t} = -G \quad (15)$$

The differential Eq. (15) is solved using the two-stage Runge-Kutta method. This leads to the subsequent explicit procedure:

1) *Update with Source Term by $\Delta t / 2$:*

$$\bar{U}_i^{j+1} = U_i^{j+1} - \frac{\Delta t}{2} G(U_i^{j+1}) \quad (16)$$

2) *Re-update with Source Term by Δt :*

$$\bar{\bar{U}}_i^{j+1} = \bar{U}_i^{j+1} - G(\bar{U}_i^{j+1}) \Delta t \quad (17)$$

The resulting value of $\bar{\bar{U}}_i^{j+1}$ is then taken as the solution at time $j+1$.

VIII. RESULTS AND DISCUSSION

The objective of this section is to compare and evaluate the accuracy and efficiency of the MacCormack adapted scheme with the artificial viscosity and the fractional technique for solving water hammer problems in a friction pipe. The analyzed system, shown in Fig. 3 consists of a single pipe with length $L = 10 \text{ km}$, wave speed $a = 1000 \text{ m/s}$ and diameter $D = 1 \text{ m}$. The steady state flow $Q = 2 \text{ m}^3/\text{s}$, under a static reservoir head $H = 400 \text{ m}$. With a roughness of 1 mm , the resulting Darcy-Weisbach friction factor, calculated using the explicit formula proposed by Achour [15] is $\lambda = 0,01976$.

The transient conditions were produced by instantaneously closing the downstream valve at $t = 0$. The pipe was divided into 30 reaches. With Courant number $C_r = 1$, the time step is then $\Delta t = 0,333 \text{ s}$.

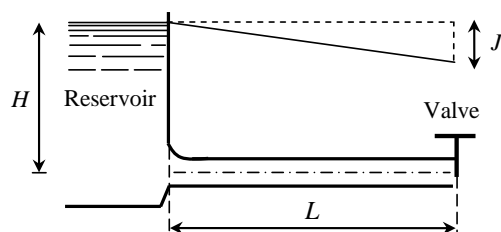


Fig. 3 Piping system

The Fig. 4 shows the pressure variation at the valve end with time, calculated by the original MacCormack scheme. It can be seen that the numerical solution obtained presents numerical instability, which appears beyond the first half period of the transient phenomenon.

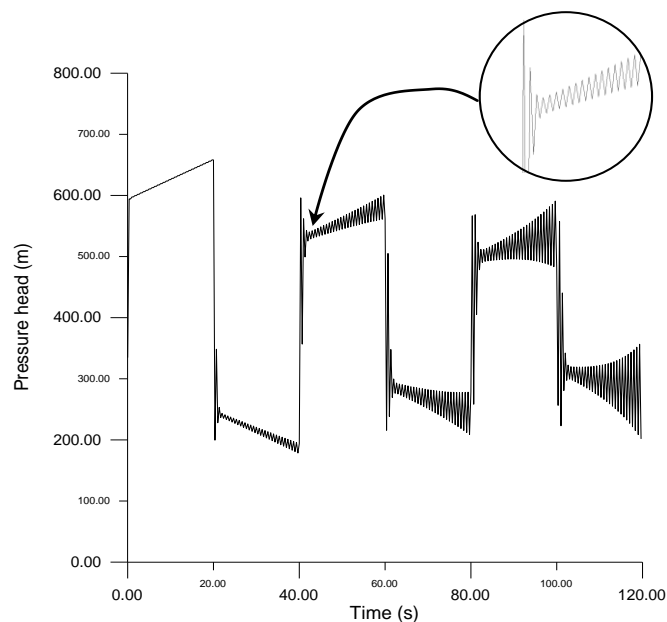


Fig. 4 Pressure head variation at the valve

These strong numerical oscillations affect the monotony of the solution and are accentuated with time until total divergence of the MacCormack scheme.

On Fig. 5, one can note as a result of the numerical instability, that the shock is captured with spurious oscillations, growing with time, on the backside of the wave front. The profiles shown are between time 42 s, 45 s and 48 s.

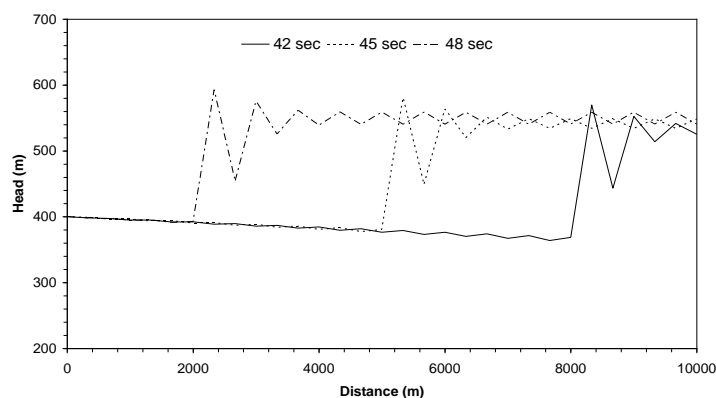


Fig. 5 Wave front advancing (without artificial viscosity)

Following the technique used by Fennema and Chaudhry [7], these dispersive errors introduced herein, are reduced by adding dissipative terms of the same or higher order than the scheme itself. A plot showing the influence of the artificial viscosity terms on the pressure diagram at the downstream valve is ported on Fig. 6.

A comparison between Fig. 4 and Fig. 6 allows to conclude that the artificial viscosity, with an optimal dissipative value $\Phi = 0,4$, eliminates the spurious oscillations and corrects thus the numerical solution of the MacCormack scheme.

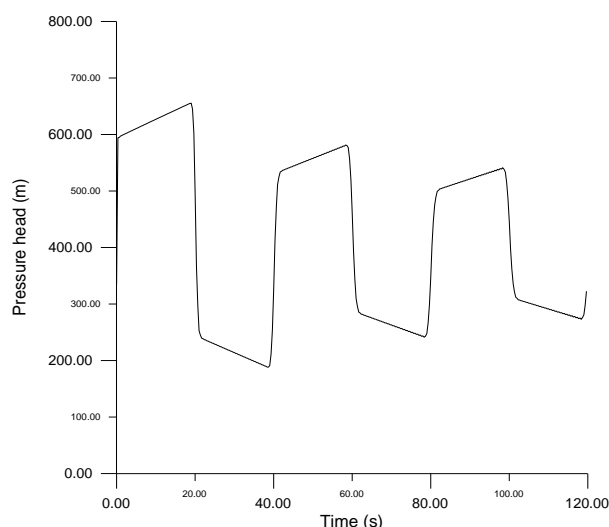


Fig. 6 Pressure head variation at the valve (with artificial viscosity)

One can see in the Fig. 7, however that the artificial viscosity, which smoothes the wiggles (Fig. 5), leads to smearing the wave front. The shock is hence not correctly captured and the solution becomes excessively dissipative. In this case, the shape of the traveling wave is not a true representation of the physical phenomenon being simulated. It must be steep because of the abrupt valve operation.

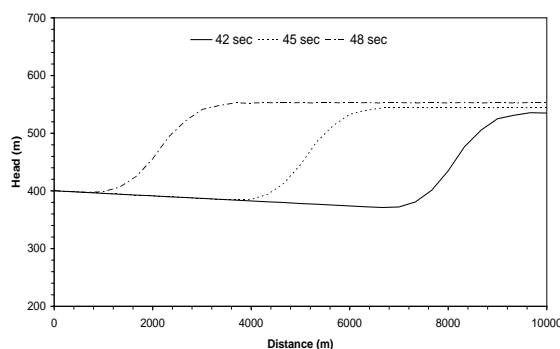


Fig. 7 Wave front advancing (artificial viscosity added)

The application of the fractional (splitting) method for the MacCormack scheme for this problem leads to better results as shown in Fig. 8. Comparing the resulting diagram with those obtained previously, it follows that none dispersion or diffusion appears. Though the maximum pressure head is relatively the same one calculated with the fractional technique ($658,99\text{ m}$) and the original MacCormack scheme ($658,34\text{ m}$), the minimum pressure is however different with a notable error. The cause is due to the error accumulation in the original scheme before his complete divergence.

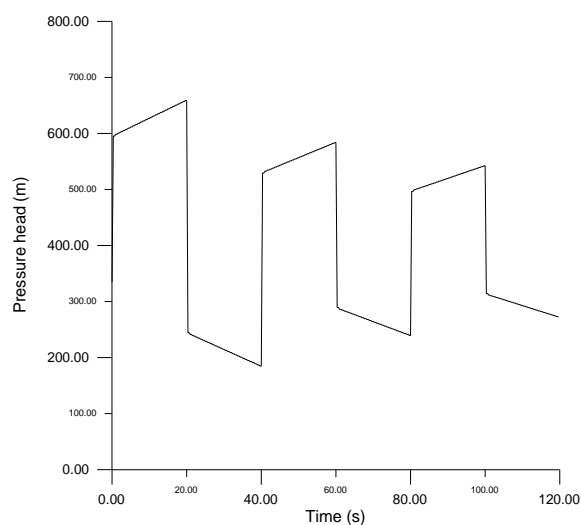


Fig. 8 Pressure head variation at the valve (splitting technique)

One can note that the artificial viscosity not only smoothes the wave front but also reduces the extreme pressures rising from the sudden closure of the valve. Indeed, the peak of the pressure computed using the last procedure is $655,55\text{ m}$, i.e. a difference of $3,44\text{ m}$. During the down surge, the minimal value computed is $188,36\text{ m}$, a difference also of $3,44\text{ m}$ compared with value obtained by the fractional technique ($184,92\text{ m}$).

The Fig. 9 illustrates the shock wave traveling upstream at time 42 s , 45 s and 48 s . It appears then clearly that the shock is well captured without any spurious oscillations or diffusion in the computed solution.

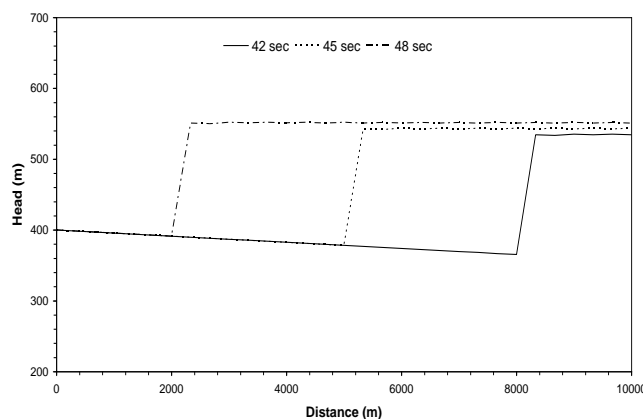


Fig. 9 Wave front advancing (splitting technique)

It can be seen in the Fig. 9 the process of upward adjustment reported in [16]. This process consists in the increasing of the pressure head at each point as the surge wave propagates upstream. The physical phenomenon is then accurately reproduced herein.

The Fig. 10 gives a numerical synthesis of the phenomenon evolution in the space-time plane.

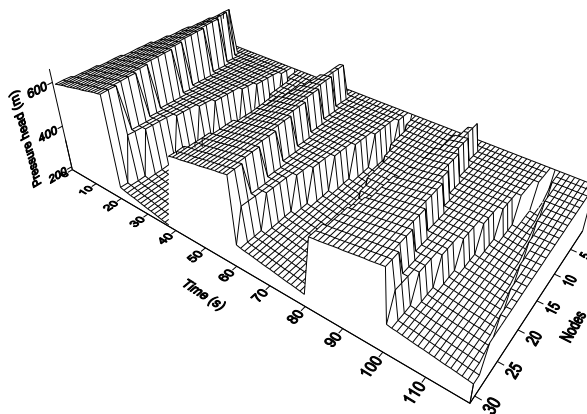


Fig. 10 Perspective view of the wave propagation in space-time plane

IX. CONCLUSION

The MacCormack finite-differences scheme has been successfully used to solve hyperbolic systems, such as unsteady free surface flow. However, its application to water hammer problems received little attention in literature. In the present paper the implementation of the improved MacCormack scheme for transient analysis of a friction pipelines is presented. It was shown that the original unsplit MacCormack scheme leads to numerical instability and spurious oscillation near steep gradient. The introduction of an artificial viscosity deteriorates the numerical solution because of the smearing the shock wave front. Comparison of the computed results with the adapted MacCormack scheme, using the fractional step method, has led to better results without any diffusion or dispersion errors. It seems therefore that the proposed technique can be promising for the practical water hammer simulation.

REFERENCES

- [1] M. H. Chaudhry, "Applied hydraulic transients," Van Nostrand, 1979.
- [2] B. De Almeida and E. Koelle, "Fluid transients in pipe networks," Computational Mechanics Publications, Elsevier Applied Science, Glasgow, 1992.
- [3] E. B. Wylie and V. L. Streeter, "Fluid transients," MacGraw-Hill, 1978.

- [4] M. H. Chaudhry and M. Y. Hussaini, "Second-Order Accurate Explicit Finite-Difference Schemes for Water Hammer Analysis," ASME J. Fluids Eng., vol. 107, pp. 523-529, 1985.
- [5] M. S. Ghidaoui, M. Zhao, D. A. McInnis., and D. H. Axworth, "A review of Water hammer Theory and Practice," Transactions of the ASME, vol. 58, pp. 49-76, 2005.
- [6] R. W. MacCormack, "Numerical solution of the interaction of a shock wave with a laminar boundary layer," Lectures notes in physics, Springer-Verlag, vol. 8, pp. 151-163, 1971.
- [7] R. J. Fennema and M. H. Chaudhry, "Explicit Numerical Schemes for Unsteady Free-Surface Flows with Shocks," Water Resources Research, vol. 22, no. 13, pp. 1923-1930, 1986.
- [8] R. Garcia and R.A. Khawawita, "Numerical solution of the St Venant equations with the MacCormack finite-differences scheme," International Journal for Numerical Methods in Fluids, no. 6, pp. 259-274, 1986.
- [9] A Berreksi, A. Kettab, B. Remini, and S. Benmamar, "Computation of two-dimensional unsteady supercritical flows in open channel contraction of spillway chutes," Dam Engineering-International Water Power and Dam Construction, vol. XIX, iss. 3, pp. 149-168, 2008.
- [10] M. H. Chaudhry, "Open Channels Flow," Second edition, Springer, 2008.
- [11] S. M. Bhallamudi and M. H Chaudhry, "Computation of flows in open-channel transitions," Journal of Hydraulic Research, vol. 30, no. 1, pp. 77-93, 1992.
- [12] M.B. Abbott, "An introduction to the method of characteristics," Thames and Hudson, 1966.
- [13] A Berreksi, "Modélisation et simulation numérique des écoulements non permanents dans des canaux à ciel ouvert," PhD thesis in Hydraulics sciences, Polytechnic National School of Algiers, 2012.
- [14] R. J. Leveque, "Finite Volume Methods for Hyperbolic Problems," Cambridge University Press, Cambridge, 2002.
- [15] B. Achour and A. Bedjaoui, "Tubulent Pipe-flow Computation Using the Rough Model Method," International Journal of Civil Engineering and Science, vol. 1, no.1, pp. 36-41, 2012.
- [16] B. E. Larock, R. W Jeppson ,and G. Z. Watters, "Hydraulics of pipeline systems," CRS Press, 2000.