

# Rolling Analysis in Diffusion Processes

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**Abstract-**The model stability is an important condition, when analyzing a financial time series. This assumption may be violated because of changes in economical and financial conditions. In a parametric model, the model stability assumption is reduced to constancy of parameters of model. Rolling analysis is useful tool to detect changes in the parameters of a statistical model. This paper considers the rolling analysis for diffusion processes. Theoretical aspects are proposed and some simulation examples are given. A real data set is considered. A conclusion section is also given.

**Keywords-**Diffusion process; Financial time series; Model stability; Rolling analysis

## I. INTRODUCTION

It is necessary for financial time series to check their model stability over time. For example, the volatility of a portfolio may increase as well as risk premium rises. This concept is also a key assumption for prediction. In a parametric model, the model stability condition is reduced to model parameters constancy. However, economical conditions in real world often change and it isn't reasonable to assume that parameters of a model remain unchanged. A method to check the constancy of parameters is rolling analysis. In this method, over a rolling window of sample, parameters estimates are computed. When parameters are constant over the sample, these estimates are very close to each other. However, if parameters change at some time point, then the rolling estimates capture these instabilities. Some excellent references in this field are Colby and Meyers (1988), Diebold and Mariano (1995), and Alexander (2001). Zivot and Wang (2008) proposed some *S+FinMetrics* functions for rolling analysis in financial time series.

Although rolling analysis has been received considerable attentions in discrete time stochastic process, it seems there is no existing literature for continuous time series such as the diffusion process. This series is useful tool in many applications such as economic, finance and insurance, see Kutoyants (2004). Therefore, in the current paper, we consider the problem of rolling analysis in diffusion processes. The current paper is organized as follows. In section 2, we propose theoretical aspects of rolling analysis for a diffusion process. Some simulation results and application in a real data set are considered in section 3. Conclusions are given in section 4.

## II. ROLLING ANALYSIS

In this section, we prepare the theoretical foundation of rolling analysis for diffusion process by presenting some simple examples.

Example 1. Let  $X_t$  be a Brownian motion with a drift defined by

$$dX_t = \mu dt + \sigma dW_t, t \in [0, T]$$

where  $X_t$  is standard Brownian motion on  $[0, T]$ . This model is one of the first candid for modeling the price

movement of asset prices. It is clear that we ask if the parameters say  $\mu$  is constant as time passes. Following Alexander (2001), a  $h$ -period (for some  $h > 0$ ) simple moving average is given by

$$\hat{\mu}_t(h) = \left(\frac{1}{h}\right) \int_0^h X_{t-s}' ds = \frac{X_t - X_{t-h}}{h}.$$

The above quantity is good tool for detecting shift in parameter  $\mu$ . The standard error band around the rolling estimate of  $\mu$  is computed as  $\frac{\sigma^2}{h}$ . A 95% confidence interval for  $\mu$  based on rolling estimate is

$$\hat{\mu}_t(h) \pm 1.96 \frac{\sigma}{\sqrt{h}}.$$

In practice, it is not possible to plot  $\hat{\mu}_t(h)$  for every  $t$  in  $[0, T]$ . Therefore, we apply this method for some predefined time points  $t_i, i = 1, \dots, n$  and we use a discrete approach such as Euler method. In application,  $\sigma$  is unknown and it is replaced by its estimate namely  $\left(\frac{1}{n}\right) \sum_{i=1}^n (X_{t_i} - \bar{X})^2$ .

Example 2. Let  $X_t$  be a geometric Brownian motion defined by

$$dX_t = \theta X_t dt + \sigma X_t dW_t$$

This model is studied by Black and Scholes (1973) for option pricing formula. This formula had a great influence on financial mathematics. Naturally, changes in  $\theta$  will destroy the obtained inferential results if this instability is not detected. To check the stability of  $\theta$ , it is needed to compute and monitor the plot of

$$\hat{\theta}_t(h) = \left(\frac{1}{h}\right) \int_{t-h}^t X_s^{-1} dX_s$$

A discrete approximation for above quantity is

$$\left(\frac{1}{n_h}\right) \sum_{\{i; t-h < t_i < t\}} \left(\frac{X_{t_i} - X_{t_{i-1}}}{X_{t_{i-1}}}\right).$$

where  $n_h$  is the number of  $t_i$ 's belonging to interval  $(t, t+h)$  and  $X_{t_i}$  is Euler (or any suitable method) approximation of  $X_t$ . The standard error and 95 percent confidence band are  $\frac{\sigma}{\sqrt{h}}$  and  $\hat{\theta}_t(h) \pm 1.96 \frac{\sigma}{\sqrt{h}}$ , respectively.

Example 3. The option pricing formula of Black-Scholes assumes that the volatility parameter is constant. However, a stochastic volatility model is fitted much better to financial time series. Hull and white (1987) proposed the following model

$$\begin{cases} dX_t = \theta X_t dt + \sigma_t X_t dW_t \\ d\sigma_t^2 = \mu \sigma_t^2 dt + \xi \sigma_t^2 dB_t \end{cases}$$

where  $W_t$  and  $B_t$  are two standard Wiener processes such that  $\text{cor}(dW_t, dB_t) = \rho$ . Let  $v_t(h) = \int_{t-h}^t -\sigma_s^{-2} ds$ . In this case, we should study the behavior of the following rolling weighted mean

$$\hat{\theta}_t(h) = (v_t(h))^{-1} \left\{ \int_{t-h}^t \sigma_s^{-2} \frac{dX_s}{X_s} \right\}$$

and therefore, we have

$$E(\hat{\theta}_t(h)) = \theta \text{ and } \text{var}(\hat{\theta}_t(h)) = E((v_t(h))^{-1})$$

It's easy to see that given  $v_t(h)$  the  $\hat{\theta}_t(h)$  is normal with zero mean and conditional variance  $(v_t(h))^{-1}$ . A conditional 95% confidence interval for  $\theta$ , given  $v_t(h)$  is  $\hat{\theta}_t(h) \pm \frac{1.96}{\sqrt{v_t(h)}}$

Example 4. Let  $X_t$  be an Ito diffusion process indexed by parameter  $\theta$  satisfying in the following stochastic differential equation (SDE)

$$dX_t = a(X_t; \theta)dt + b(X_t)dW_t$$

Here, we assume that there is a unique solution for this SDE, see Kutoyants (2004). Let  $\hat{\theta}_{t,h}$  be the maximum likelihood of  $\theta$  based on observations  $X_s$ ,  $s \in [t-h, t]$ . Following Kutoyants (2004), as  $h \rightarrow \infty$ , the estimator  $\hat{\theta}_{t,h}$  is consistent estimator for  $\theta$  and it is also unbiased. Therefore, we can use  $\hat{\theta}_{t,h}$  as a shift detector. Also, this estimator is asymptotically normal for each fixed  $t$ , therefore, we can construct an asymptotically confidence bands. If  $\theta$  was a vector of unknown parameters again, the above method works. For example, in Black-Scholes model (example 2), using the rolling maximum likelihood of  $(\theta, \sigma)$  we can detect the changes in two parameters, simultaneously.

Example 5. In example 4, suppose that the diffusion coefficient  $\sigma$  may also relate to unknown parameter  $\theta$ . In this case, it is enough to apply the Ait-Sahalia (2002) transform

$$Y_t = \gamma(X_t; \theta) = \int^{X_t} \frac{du}{b(u; \theta)}$$

To derive a SDE with unit diffusion coefficients in the form of

$$dY_t = a_Y(Y; \theta)dt + dW_t$$

and again we apply the approach of example 4. Here,

$$dY_t = \frac{a(\gamma^{-1}(y; \theta); \theta)}{b(\gamma^{-1}(y; \theta); \theta)} - \frac{1}{2} \frac{\partial}{\partial x} a(\gamma^{-1}(y; \theta); \theta).$$

For example, consider the SDE

$$dX_t = \theta X_t dt + \theta^2 X_t dW_t$$

$\gamma(y; \theta) = \int^y \frac{du}{\theta^2 u} = \frac{1}{\theta^2} \ln(y)$ ,  $\frac{a(x; \theta)}{b(x; \theta)} = \frac{1}{\theta}$  and  $\frac{\partial}{\partial x} a(x; \theta) = \theta$ . Therefore,  $a_Y(y; \theta) = \frac{2-\theta}{\theta^2}$  and  $dY_t = \frac{2-\theta}{\theta^2} dt + dW_t$ . Again, approach of example 1 is applicable here.

### III. SIMULATIONS

In this section, we propose some simulation results and a real data set are studied.

#### A. Simulations.

In this sub-section, we present some simulation results and a real data set analysis to evaluate the performance of rolling method appeared in section 2. We refer to simulation schemes in Zivot and Wang (2008). We consider two cases, (a) the Black-Scholes model under the stochastic volatility assumption and (b) monitoring the stability of drift and diffusion coefficients of a SDE, simultaneously.

Case (a). Consider example 2,

$$dX_t = \theta X_t dt + \sigma X_t dW_t, 0 < t < 1.$$

First, we let  $X_0=1, \theta=0, \sigma=1, t_i = \frac{i}{400}, i=1, \dots, 400$  and

$n_h = 24$ . We also compute returns  $r_i = \frac{X_{t_i} - X_{t_{i-1}}}{X_{t_{i-1}}}$ . The

return series  $r_i$  is plotted in Figure 1 (page 7). The standard error of  $\hat{\theta}_{t_i}(h)$  and  $\hat{\sigma}_{t_i}(h)$  are approximated by

$$\frac{\hat{\theta}_{t_i}(h)}{\sqrt{n_h}} \text{ and } \frac{\hat{\sigma}_{t_i}(h)}{\sqrt{2n_h}},$$

respectively. Figure 2 (page 7) shows the rolling estimates of  $\hat{\theta}_{t_i}(h)$  with 95% confidence intervals. Note that since we are monitoring  $\theta$  and  $\sigma$  simultaneously, to avoid excess of error, according to the Bonferroni law, we obtain the marginal intervals for each parameter in significant level 97.5%. The rolling drifts and diffusions clearly vary over the sample. The rolling drifts start out around 0.1%, fall close to 0% very soon, rise again a little, but we can assume that it is zero for the entire sample. The plot of  $\hat{\sigma}_{t_i}(h)$  is also derived (which isn't given here). The rolling diffusion values start out around 1 and remain unchanged for all sample points. These facts

motivate us to assume that the parameters are constant over the sample. Now, suppose that a change has occurred in  $\theta$  and it equals to 0 until  $t_{200}$  and shifts to 0.7 after  $t_{200}$ . Then the plot of  $\hat{\theta}_t(h)$  (Figure 3, page 7) obviously shows that there is a shift in parameter  $\theta$ .

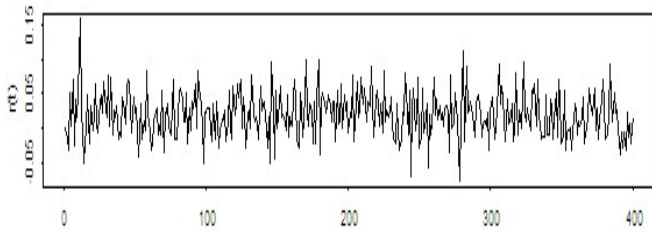


Figure1. Return of B-S series: no change

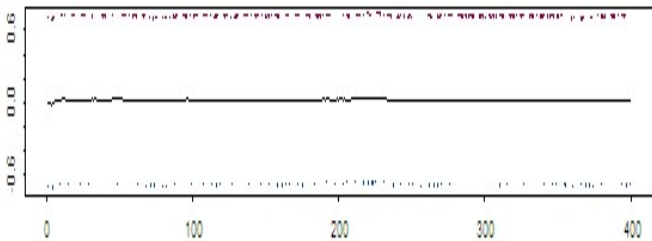


Figure2. Rolling means with confidence bands

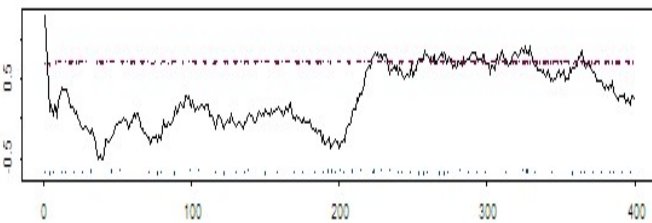


Figure3. Rolling means with a change point

Case (b). In example 3, we derived the conditional confidence interval using rolling drift estimates. However, in real situations, observations sampled from the stochastic volatility are not fixed. To avoid this difficulty, we should derive the quantile of marginal distribution of  $\hat{\theta}_t(h), t \in [0, 1]$  i.e.,  $F(z)$

$$F(z) = E\{\Phi(\sqrt{v_t(h)}(z - \theta))\},$$

where  $\Phi(\cdot)$  is the cdf of standard normal distribution. For example, for  $\theta = 0.3$ ,  $\mu = 0.4$ ,  $\xi = 1$ , for  $t = 0.5$  and  $h = 0.05$ , Monte Carlo estimate of  $F(z)$  with  $M = 1000$  repetitions is 0.45. Related standard deviation of Monte Carlo estimate is 0.0157. The 0.95-th quantile of  $\hat{\theta}_t(h)$  is the solution of

$$E\{\Phi(\sqrt{v_t(h)}(q_{0.95} - \theta))\} = 0.05$$

A Newton-Raphson method is applied here as

$$q_{0.95}^k = q_{0.95}^{k-1} - \frac{E\{\Phi(\sqrt{v_t(h)}(q_{0.95}^{k-1} - \theta))\}}{E\{\sqrt{v_t(h)}\phi(\sqrt{v_t(h)}(q_{0.95}^{k-1} - \theta))\}}$$

where  $\phi(\cdot)$  is the density of standard normal distribution. Using a proper numerical software, we found that  $q_{0.95} = 3.5$ .

## B. Real Data Set

The diffusion process has many applications in modeling the economical and financial time series. In this way, Hansen (1992) considered the annual U.S. output growth rates over time period 1889- 1987. He fitted a discrete time AR(1) model for this series. He examined many goodness of fit tests and concluded that his model worked very well for the data set in hand. He also derived many prediction and concluded that again this model works very well. He showed that this model has remained stable. Therefore, there is no change point in this model. However, since every discrete time series is an approximation of a continuous time series, following Hansen (1992), we also fitted a continuous time AR(1) (as alternative and approximation method) to this series in the form of

$$dX_t = \alpha X_t dt + \sigma dW_t.$$

We found that  $\hat{\alpha} = -0.68$  and  $\hat{\sigma} = 0.045$ . We applied the above mentioned method for this data set and we understood that the maximum of absolute of difference of rolling drifts and  $\alpha$  is less than 0.1. This shows that the parameter  $\alpha$  remains unchanged during period of study. Therefore, since the continuous time series has remained unchanged, the discrete time series has also no change point. We also proposed many goodness of fit tests, where their results were not given in this paper.

## IV. CONCLUSION

Diffusion processes play important role in statistical inference and the model stability is a necessary assumption for them. Rolling estimates monitor the constancy of parameters of a model over time. In this paper, we considered the rolling analysis in a diffusion process. Theoretical aspects are given and by some simulations, we show that these estimates detect changes very well. We proposed some examples and a real data set. Our real data set is Hansen's data set about annual U.S. output growth rates over time period 1889- 1987. We also understood that the series remains unchanged during the mentioned period of time.

## REFERENCES

- [1] Y. Ait-Sahalia, "Maximum likelihood estimation of discretely sampled diffusions: a closed form approximation approach". *Econometrica*, 223-262, 2002, vol 70.
- [2] Alexander. *Market Models: A Guide to Financial Data Analysis*. John Wiley & Sons, 2001.
- [3] Black and M. Scholes, "The pricing of options and corporate liabilities". *Journal of Political Economy*, 635-654, 1973, vol 81.
- [4] R. W. Colby and T. A. Meyers, *The Encyclopedia of Technical Market Indicators*. McGraw-Hill, 1988.
- [5] X. Diebold, and R. S. Mariano, "Comparing Predictive Accuracy". *Journal of Business and Economic Statistics*, 253-263, 1995, vol 13.
- [6] B. E. Hansen, "Testing for parameter instability in linear models", *Journal of Policy Modeling*, 517-533, 1992, vol 14.
- [7] Hull and A. White, "The pricing of options on assets with stochastic volatilities". *Journal of Finance*, 281-300, 1987, vol 42.
- [8] Y. A. Kutoyants, *Statistical Inference for Ergodic Diffusion Processes*. Springer, 2004.
- [9] E. Zivot and J. Wang, *Modeling Financial Time Series with S-PLUS*. Springer, 2008.