Estimating Project Makespan for Multi-critical Paths Networks

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Abstract- **In this study,we concentrate on a very common project management problem, called *estimatingthe project makespan*. The literature usually uses the Central Limit theorem to estimate the project makespan. Here, we show that in multi-critical paths projects, the project's makespan behaves differently with the regular estimates.**

We analyzed project makespan using both the analytic approach and simulation. Our results show that, in many cases, it is biased to use the critical path length when attempting to estimate project makespan.

The current study presents the distribution of the project makespan for cases where there are several critical paths. It shows the results for common probability distributions of the critical path duration. The results of the current study provide users with better estimates and upper bound in regard to project completion time.

Keywords- PERT; Central Limit Theorem; Project Makespan; Simulation; Project Completion Time

I. INTRODUCTION

In this study, we concentrate on a very common project management problem, called *estimatingthe project makespan*. The literature usually utilizes the Central Limit theorem to estimate the project makespan. Project makespan is defined as *the duration of the critical path.* In most cases, after the critical path has been defined, the project makespan is estimated by the average of the sum of the activities' duration along the critical path.

The drawback of this approach is that it ignores cases in which other paths from beginning to the end, which are not critical, last longer than the critical path, due to stochastic effects. As we will show in this study, this case is not only theoretical, but can also be very practical.

In most of the Project Management books, project makespan is presented as the length of the critical path (see Bruke [3], Winston [16],Wysocki et al. [17], Meredith & Mantel [12], and Stevenson [15]). The authors assume that all the conditions of the central limit theorem hold true. In this case, the length of the critical path that approximates the project makespan can be calculated by the sum of its activities. This means that the project makespan is estimated by the length of the critical path. Jin et al. [9], who studied the permutation flowshop schedules, empasized that "The makespan distribution of permutation flowshop schedules has been a topic of debate for almost fifty years. Many researchers have confirmed or doubted the famous claim that the makespan distribution of permutation flowshop schedules is asymptotically normal if the number of jobs is sufficiently large".

In reality, there are many projects in which there are several critical paths. In order to shorten the project's makespan, we should analyze the critical paths and try to shorten each one individually. According to CPM methods, we should shorten the longest path and then repeat this process. However, in the end, there may be several paths of equal length, all of which are critical. In this case, and in order to estimate the project'smakespan time, we should look for the maximum path's length among all the critical paths. In this study, we focus on project makespan for cases in which there are several critical paths or paths that are perceived as being "close" to critical paths (a path is "close" to a critical path when its duration is close to the critical path's duration). The measurement of a close path is determined according to the standard deviation of the path's duration.

Many studies have been published on the estimation of project makespan. Dodin [5] presented some lower and upper bounds to a project's makespan by reducing the project's network into a single equivalent activity, which starts at the beginning and ends with the project's conclusion. A very important result of this work is that it provides decision-makers with an upper bound to the project's makespan. Unfortunately, this work does not present any estimate on the distance of the upper bound and the project's span time. Schmidt et al. [13] present an analytic approach to the project makespan, when the duration distributions are PD-type functions. Their approach is to combine, at each step, two consecutive activivities into one, by either taking the maximum time, if they are parallel or the sum of both, if they are consecutive. Many algorithms were developed for scheduling problems and software packages with and without uncertainty. Since the scheduling problem is extremely important, but analytically very complex, many heuristic algorithms were developed for specific scheduling problems.

Herroelen and Leus [8] provided a review of the procedures that deal with uncertainty in project scheduling, in an attempt to respond as effectively as possible to the realized situations. Other reviews and classifiations presented by Framinan et al. [6]

focus on scheduling permutation flow-shop problems. Kolisch & Hartmann [10] updated a computational comparison review among different heuristic algorithms for scheduling problems with resource constraints.

A new study by Bruni et al. [4] emphasizes the well-known phenomenon that scheduling based on deterministic activities' duration, even without resource constraints, usually causes underestimation of the project makespan. They present a method for obtaining relevant information about the project makespan, with dependent random processing time available in the form of scenarios.

Another approach is presented by Abdelkader [1], for cases in which the activities' durations are Weibull-distributed. In his paper, he uses the moment method to present some bounds on the project's makespan. However, once again, the author does not present the distance between the bound and the real project makespan.

A special case of evaluation regarding project makespan, when the activities' duration times are Erlang-distributed, is presented by Bendell et al. [2] . They show the results in five evaluation methods:

1. An analytical approach, which is usually very limited, due to calculation difficulties;
2. A numerical calculation comprised of the convolution of all the time distributions of the activities participating in the critical path;
3. Moment methods that reduce the project network into a single network arc, on which it uses four first moments in evaluating the project makespan;
4. PERT analyses;
5. Simulation of all the activities' duration distributions.

The uncertainty of activities' duration times and the consequence to project makespan are analyzed by Shtub et al. [14]. They present different projects' makespans in different simulation runs and reveal the fact that the makespan is not just the length of the critical path. They define the *Criticality Index (CI)* of an activity as 'the proportion of runs in which the activity was on a critical path'. They also refer to dependency among activities (that might use the same resource or other causes of dependencies). Theirs is one of the first books that present project makespan as a distribution function of all the paths' duration distributions. Furthermore, they also showed that if there are n independent paths $X_1, X_2, \ldots, X_n$, from a project's start to end, then the project makespan is $X=\mathrm{Max}\{X_1, X_2, \ldots, X_n\}$ and satisfies:

$$F(\tau)=P(X\le\tau)=P(X_1\le\tau)\cdot P(X_2\le\tau)\cdot\ldots\cdot P(X_n\le\tau)=\mathrm{F}_1(\tau)\,\mathrm{F}_2(\tau)\ldots\mathrm{F}_n(\tau) \quad (1)$$

where F_i is the probability distribution of the random variable X_i, and all the X_i are independent random variables.

Our aim is to estimate $\mathrm{E}(X)$, when $X \sim \mathrm{F}(x)= \mathrm{F1}(x)\cdot\mathrm{F2}(x)\ldots \mathrm{Fn}(x)$.

II. THE PROBLEM

We assume that our project was analyzed according to the CPM method and that activities were shortened whenever possible. After shortening all the activities, we then assume that the project has several critical paths. Let $X_1, X_2, \ldots, X_n$ be the duration length of all the paths in the project (not only the critical paths). $X_1, X_2, \ldots, X_n$ are random variables. If we denote the project's makespan by X, then $X=\mathrm{Max}\{X_1, X_2, \ldots, X_n\}$.

Let $\mathrm{F}_i(x)$ denote the probability distribution of the random variable X_i and $F(n,x)$ denote the probability distribution of the random variable X.

We define the point $x_{0.9}$ that satisfies $P(X<x_{0.9})=0.9$ as *Ninety Percent* (NP).

We will use the truncated distribution notation for those probability distributions with finite support. Thus, there is a real (finite) number y, such that $P(Y\le y)=1$ and $P(Y\le y)=0$, as we know the uniform distribution on [a,b] is truncated, while the normal or exponential distributions are not.

Gonen [7] showed that if the random variables $X_1, X_2, \ldots, X_n$ are truncated, mutually independent with identical truncated distribution $\mathrm{F}_i(x)$, and if $b=\min\{x/\ F_i(x)=1\}$(b exists, since the random variables are truncated), then $F(n,x)$ tends to $\delta(b,x)=\begin{cases}1 & x\ge b\\ 0 & x<b\end{cases}$, when n tends to infinity and under these conditions, $E(X)$ tends to b.When we apply the above to n independent random variables $X_1, X_2, \ldots, X_n$ that distributes U(a,b), it can be seen that $x_{0.9}=\sqrt[n]{0.9}(b-a)+a \xrightarrow[n\to\infty]{} b$.

III. SIMULATION RESULTS

In order to expand and apply the results to other cases of activities' duration distributions, we ran simulations of projects with several critical paths when all the activities' duration distributions were the same. When conducting these simulations, we assumed that there are K different critical paths that are independent and identically distributed. The duration of each critical path is a random variable X_j $j=1,\ldots K$; we examined the maximum X of these K critical paths.

The analysis was conducted on different probability distributions of X_j, and was tested N times (2,000). The statistical goodness-of-fit test was used on the simulation results of the maximum distribution. Finally, some sensitivity analyses were added to the number of critical paths(K) parameter.

The following notations and abbreviations will be used throughout this work:

The *original distribution* is the probability distribution of the critical path's length X_j.

The *maximum distribution* refers to the probability distribution of X=Max(X_1, X_2, …,X_K), and X is a random variable called the *maximum random variable*.

The project makespan is calculated by summing up the lengths of the critical paths' arcs. We define the measurement criterion in order to compare the different alternatives.

The measurement criterion used is as follows:

$$MC = \frac{\text{Average}\,(\text{Max}(X_1, X_2,\ldots,X_K)) \bullet \text{E}\,(X_j)}{\sigma_{X_j}} \tag{2}$$

The MC presents the difference between the "maximum" value and the expected value in the standard deviation scale.

IV. THE SIMULATION TEST PLAN

The simulation provides us with the expected makespan and the "ninety percent" bound, which guarantees a deviation of no more than ten percent.

Gonen [7] verified the above results for truncated distributions (with a bounded support) by simulation. Uniform, triangular and similar distributions were tested and the maximum converged, as expected, to the upper bound of the density function support.

In the current work, we analyze unbounded probability distributions, where all of the critical paths are Normally, Erlang, Weibull, Betaor exponentially distributed. In each test, the following calculations were made:

- The MC criterion;
- Sensitivity - the number of critical paths – K;
- The distribution of the maximum X;
- The "ninety percent" bound to the makespan, using empirical simulation results;
- F(K,X) distribution versus the theoretical"ninety percent" bound using the original distribution.

V. NORMAL DISTRIBUTION (10, 2)

The normal distribution is the most common probability distribution. In this section, we assume that the project's critical path duration is distributed according to the normal probability distribution. Actually, in most of the literature, it is assumed that the critical path duration is normally distributed, as was indicated earlier. We assumed all the critical path's duration distributions to be N(10,2).We used expectation 10, so that we would not have to truncate negative results. The following table shows, for each number of critical paths K, the average of the maximum, its standard deviation, and the criterion MC (defined at (2)).

TABLE 1 RESULTS FORTHE *K* NORMAL N(10,2) VARIANTS

K	10	20	30	40	50	60	70	80	90	100
AV(Max)	13.09	13.74	14.10	14.33	14.50	14.65	14.77	14.87	14.95	15.03
SD(Max)	1.15	1.03	0.98	0.95	0.91	0.91	0.88	0.87	0.86	0.86
MC	1.55	1.87	2.05	2.16	2.25	2.33	2.39	2.43	2.48	2.51

From the above table, it can be seen that when all the paths' durations are distributed normally, the average increases with K. Since the normal distribution is not bounded from above, the theoretical results about truncated distributions do not hold true. This did not surprise us and when we measured the MC for K=200,000, it was 4.32, which means it cannot be limited, as the normal distribution itself is not bounded. The most important result from the table above is that since the MC ranges between 1.5 and 2.5, the project makespan can increase by 250%. This result cannot be ignored when estimating project makespan.

The estimated "ninety percent" bound of the maximum when K=100 is 16.17, while the "ninety percent" bound of the normal distribution N(10,2) is 12.56. The MC, in this case, is 1.805, which shows an increase of 180%.

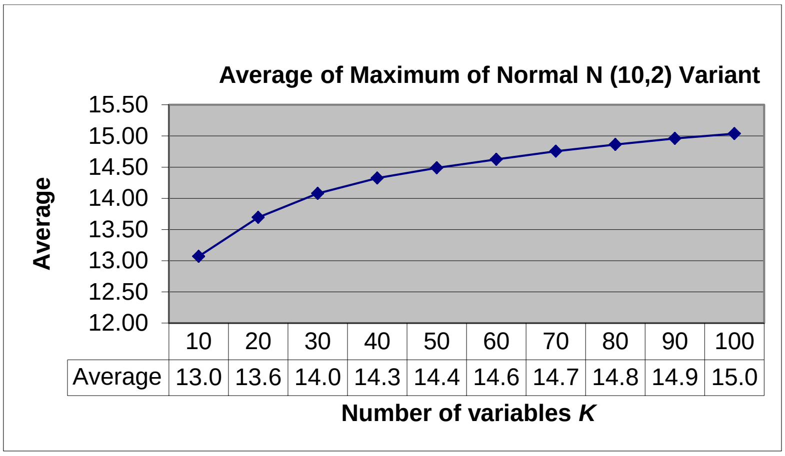
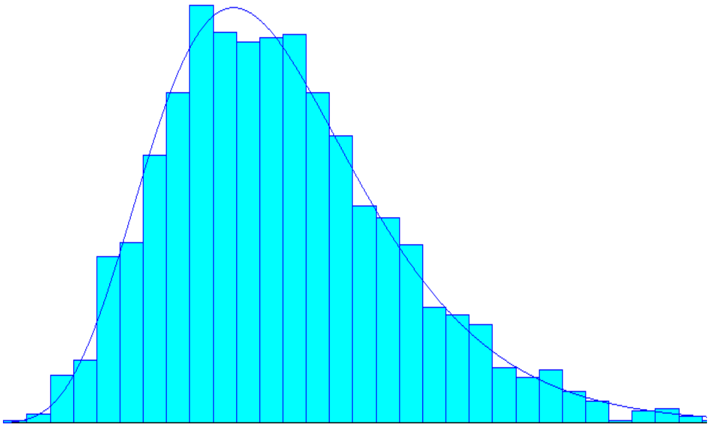


Fig. 1 Average of the maximum of the normal N(10,2) variant

Figure 1 shows that the average of the maximum converges to 2.5. However, we tested it for a much bigger *K* and found that it does not converge. Due to the importance of the normal distribution, the distribution of the maximum and its average was also tested. The following chart presents the behaviour of the maximum distribution of 100 normal deviations with an expectation of 10, a standard deviation of 2, and a sample size of 2,000 points. It can be seen that the distribution with the best fit is the Lognormal distribution, shifted by 12.1 units. The parameters are (2.92,0.891).



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        Distribution Summary

Distribution:   Lognormal
Expression:     12.1 + LOGN(2.92, 0.891)
Square Error:   0.000351

Chi Square Test
  Number of intervals   = 25
  Degrees of freedom    = 22
  Test Statistic        = 23.9
  Corresponding p-value = 0.365

Kolmogorov-Smirnov Test
  Test Statistic        = 0.0163
  Corresponding p-value > 0.15

        Data Summary

Number of Data Points   = 2002
Min Data Value          = 12.7
Max Data Value          = 18.8
Sample Mean             = 15
Sample Std Dev          = 0.866

        Histogram Summary
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Fig. 2 Distribution of the maximum of the 100 normal N(12,2) variant (sample size 2002)

For the goodness of fit test, the input analyzer of Arena simulation language was used.The results are significant according to Chi square and Kolmogorov-Smirnov tests.

Moreover, out of all other well-known probability distributions, the Erlang distribution was found to be the only one that was significant, apart from the lognormal. The following chart shows the results of the Erlang goodness-of-fit test.

```
        Distribution Summary

Distribution:   Erlang
Expression:     12.1 + ERLA(0.243, 12)
Square Error:   0.000393

Chi Square Test
  Number of intervals   = 26
  Degrees of freedom    = 23
  Test Statistic        = 35.8
  Corresponding p-value = 0.0448

Kolmogorov-Smirnov Test
  Test Statistic        = 0.0195
  Corresponding p-value > 0.15

        Data Summary

Number of Data Points   = 2002
Min Data Value          = 12.7
Max Data Value          = 18.8
Sample Mean             = 15
Sample Std Dev          = 0.866
```

Fig. 3 Fitting the Erlang distribution to a maximum of the 100 normal N(12,2) variant

The Erlang is the second-best fit. From Figure 3 above, it can be seen that it fits well according to the Kolmogorov-Smirnov test, but not according to the Chi Square test. This means that the maximum can be modelled as a sum of 12 exponential variants with an expected value of 0.243. Once again, the distribution is shifted by 12.1 units.

VI. BETA DISTRIBUTION

The Beta distribution is usually used to describe the probability distribution of activities' durations. We analyze the case where the critical path's duration distribution behaves according to the Beta rule. This fits many projects for which the paths, from beginning to end, include only a few activities and the project is conducted in a parallel mode. The Beta distribution with parameters α=4 and β=2 is an asymmetric distribution skewed to the right. It presents a phenomenon where, in many cases, activities occur late, rather than according to schedule.

The following table presents, for each number of critical paths *K*, the average of the maximum, its standard deviation, and the criteria MC.

TABLE 2 RESULTS FOR THE MAXIMUM OF THE *K* BETA(4,2) VARIANTS

K	10	20	30	40	50	60	70	80	90	100
Average	0.90	0.93	0.94	0.95	0.96	0.96	0.96	0.97	0.97	0.97
SD	0.05	0.04	0.03	0.03	0.02	0.02	0.02	0.02	0.02	0.02
MC	1.32	1.49	1.56	1.60	1.63	1.66	1.67	1.69	1.70	1.71

From Table 2, it can be seen that the average does not change too much (the average of one variant is 0.66). This is because the Beta distribution is limited by 1. The Beta distribution results were expected, since Beta belongs to the family of truncated functions, like the uniform probability distribution. An interesting phenomenon is that the distribution of the maximum also fits Beta, as can be seen in the next table. This is the only distribution whose maximum behaves like the original with different parameters. The "ninety percent" of the maximum distribution is 0.89, while the "ninety percent" of the distribution is 0.88 – almost the same. Once again, this is the result of a bounded distribution.

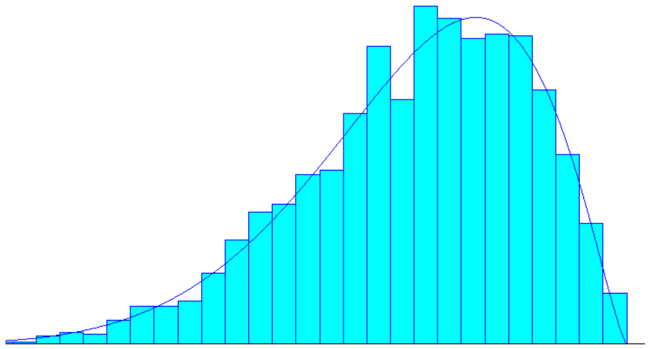

Fig. 4 Distribution of the maximum of the 100 Beta(4,2) variant (sample size 2000)

From the statistical analyses in Figure 4, it can be seen that the maximum's distribution is compressed almost to 1, with a small standard deviation of 0.016. The fitness to Beta distribution is significant. (P-value above 5%).

VII. WEIBULL DISTRIBUTION

The Weibull probability distribution has been used in previous papers, like that of Abdelkader [1]; therefore, it was added here. The Weibull distribution can be used to represent non-negative task times that are skewed to the left. This means that in many cases, the task is completed before its estimated duration. The average of the Maximum of *K* critical paths, its standard deviation, and the MC criterion are presented in the following table.

TABLE 3 RESULTS FOR THE *K* WEIBULL (2,2) VARIANTS

K	10	20	30	40	50	60	70	80	90	100
Average	3.35	3.74	3.96	4.11	4.21	4.30	4.37	4.43	4.48	4.53
S.D.	0.69	0.65	0.62	0.60	0.59	0.58	0.56	0.56	0.55	0.54
MC	1.71	2.13	2.36	2.52	2.63	2.73	2.81	2.88	2.93	2.98

The table above shows that the average of the maximum and the MC increase, together with the number of critical paths *K*. However, it does not converge, since the Weibull probability distribution is unbounded. The MC of 2.98 for *K*=100 shows a significant difference between the maximum and the original distributions.

The "ninety percent" of the maximum distribution is 5.23, while the "ninety percent" of the distribution is 3.023. As we can see, the difference is very high; therefore, it is important to use the maximum distribution instead of the original one.

The following chart shows the distribution of the maximum.

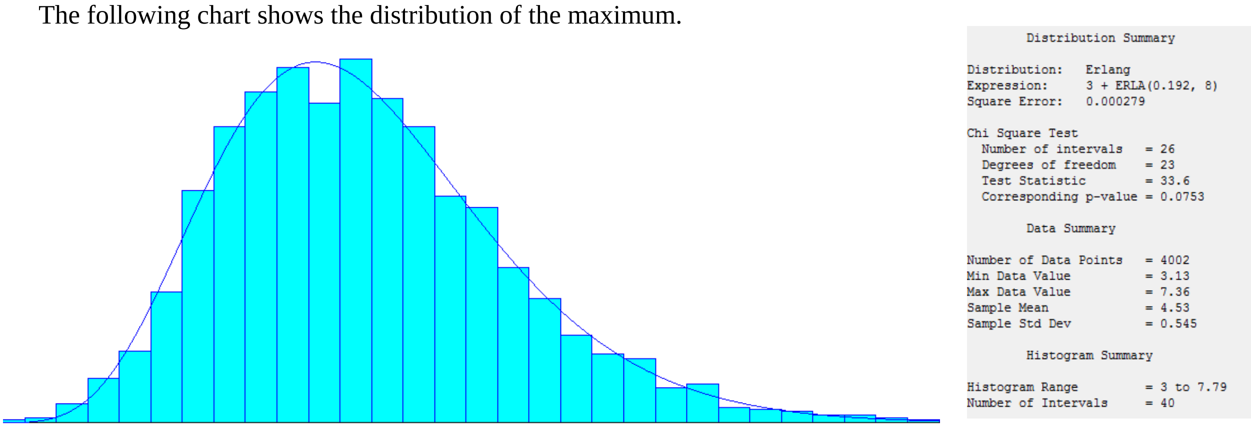

Fig. 5 Distribution of the maximum of the100 Weibull(2,2) variant (sample size 4002)

The maximum, in this case, fits the Erlang distribution. Not surprisingly, it can also fit the Gamma distribution, since the Erlang(k,β) distribution is the same as a Gamma(α,β) distribution, with the restriction that the shape parameter α must be a positive integer.

VIII. ERLANG DISTRIBUTION

The Erlang probability distribution has been used in previous papers, such as that of Bendell et al. [2]; with the shape parameter k>1, it represents an asymmetric probability distribution. The Erlang distribution can represent activities' time durations that are skewed to the right. It should be used when the task is composed of successive sub-tasks, where the time duration of each one has an exponential distribution. The following table shows the average of the maximum, its standard deviation, and the criteria MC for each number of critical paths *K*.

TABLE 4 RESULTS FOR *K* ERLANG (10,2) VARIANTS

K	10	20	30	40	50	60	70	80	90	100
Average	46.3	54.7	59.2	62.5	65.2	67.1	68.7	70.2	71.4	72.8
S.D.	15.4	15.2	15.3	15.0	14.8	14.8	14.5	14.3	14.2	14.4
MC	1.9	2.5	2.8	3.0	3.2	3.3	3.5	3.6	3.7	3.8

From the table above, it can be seen that the average of the maximum and the MC increase with *K*. Once again, it does not converge, since the Erlang probability distribution is unbounded.

The "ninety percent" of the maximum distribution for *K*=100 is 91, while the "ninety percent" of the distribution is 40 (less than half). Since the difference is very high, it is important to use the maximum distribution instead of the original one.

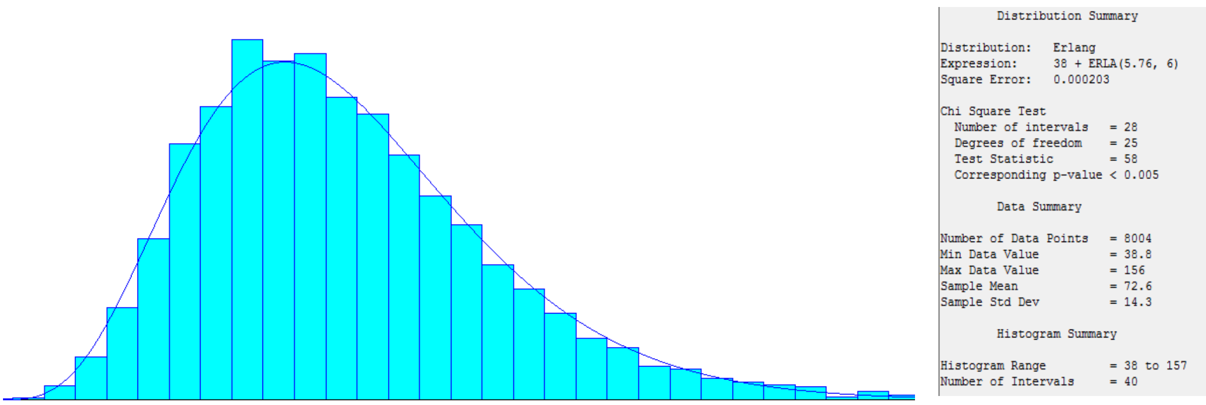

Fig. 6 Distribution of the maximum of the100 Erlang(10,2) variant (sample size 8004)

The distribution of the maximum is presented above. It behaves like the Erlang; however, it is not significantly Erlang. In the Erlang case, it is clear that the average of the maximum is significantly bigger than the original average.

IX. EXPONENTIAL DISTRIBUTION

The exponential distribution is usually used to present the time between events' appearances. The distribution may represent the completion time of tasks when they are strongly skewed to the right. Kulkarni and Adlakha [11] describe an exact method for deriving the distribution and moments of the earliest project completion time using continuous-time Markov chains (CTMCs). The following table shows for each number of critical paths *K*, the average of the Maximum, its standard deviation, and the criteria MC.

TABLE 5 RESULTS FOR *K* EXPONENTIAL (1) VARIANTS

K	10	20	30	40	50	60	70	80	90	100
AV(Max)	2.940	3.601	3.967	4.243	4.476	4.665	4.804	4.951	5.093	5.213
SD(Max)	1.265	1.225	1.236	1.223	1.248	1.241	1.241	1.265	1.276	1.288
MC	1.940	2.601	2.967	3.243	3.476	3.665	3.804	3.951	4.093	4.213

Once again, the average of the maximum and the MC increase with *K*. It does not converge, since the Exponential probability distribution is unbounded.

The "ninety percent" of the maximum distribution for *K*=100 is 6.83, while the "ninety percent" of the distribution is 2.34. As we can see, the difference is very high; therefore, it is important to use the maximum distribution instead of the original one.

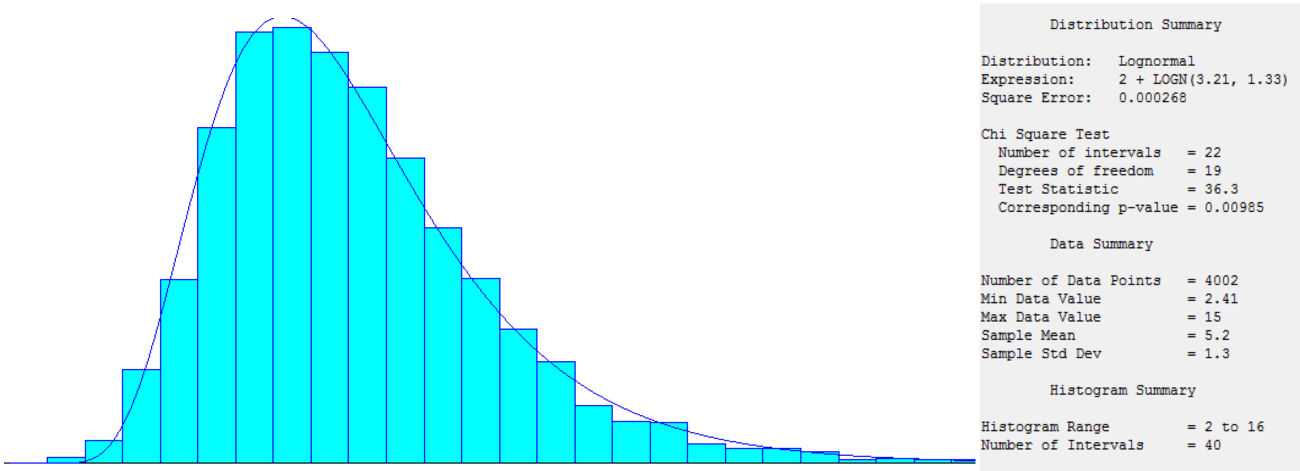

Fig. 7 Distribution of the maximum of a 100 Exponential(1) variant (sample size 4002)

The figure above shows the distribution of the maximum of a 100 exponential variant. The distribution looks lognormal, but it is not significant.

X. CONCLUSIONS

The current study presents the distribution of the project makespan for cases where there are several critical paths. The analytic approach covers only the bounded distributions. In this study, we concentrate on non-truncated (unbounded) probability distributions compared to the bounded Beta distribution.This is an expansion of Gonen's study [7], which concentrates on bounded probability distributions.

The following table shows the summary of the simulation tests for 100 critical paths:

TABLE 6 SUMMARY OF THE RESULTS OF THREE POSSIBLE PROBABILITY DISTRIBUTIONS FOR *K*=100

Probability distribution	**Average length of critical path**	**Average duration of the Maximum**	**MC**	**Ninety percent of one critical path**	**Ninety percent of the Maximum**
Normal (10,2)	10	15.03	2.51	12.56	16.17
Beta(4,2)	0.66	0.97	1.71	0.88	0.89
Weibull (2,2)	1.77	4.53	2.98	3.02	5.23
Erlang(10,2)	20	72.8	3.8	40	91
Exponential(1)	1	5.21	4.21	2.34	6.83

It can be seen that for some distributions the difference between the maximum distribution and the original cannot be ignored. Both the average of the maximum and the ninety percent are sometimes much higher than the single critical path.

The above results are very realistic; e.g., when we analyzed 4 critical paths that distributed Normally(10,1), we found that the average of the maximum is 11 (10% more than the original length).

The results show the need for further research in characterizing and estimating the project' makespan.

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