

New Exact Solutions for the VGKdV-mKdV Equation with Nonlinear Terms of Any Order

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Abstract-In this paper, some new exact solutions for the variable-coefficient generalized KdV-mKdV equation (VGKdV-mKdV) with nonlinear terms of any order are obtained by using the generalized Jacobi elliptic functions expansion method with computerized symbolic computation, some of these solutions are degenerated to soliton-like solutions and trigonometric function solutions in the limit cases, which shows that the applied method is more powerful and will be used in further works to establish more entirely new exact solutions for other kinds of nonlinear partial differential equations with nonlinear terms of any order arising in mathematical physics.

Keywords- Generalized Jacobi Elliptic Functions Expansion Method; Generalized KdV-MKdV Equation; Exact Solutions; Soliton-Like Solutions; Jacobi Elliptic Wave-Like Solutions

I. INTRODUCTION

Looking for exact solutions to nonlinear evolution equations (NEEs) has long been a major concern for both mathematicians and physicists. These solutions may well describe various phenomena in physics and other fields, such as solitons and propagation with a finite speed, and thus they may give more insight into the physical aspects of the problems. Up to now, many effective methods have been presented, such as inverse scattering transformation [1], Hirota bilinear method [2], homogeneous balance method [3], Bäcklund transformation [4], Darboux transformation [5], the extended tanh-function method [6], the extended F-expansion method [7], projective Riccati equations method [8], the Jacobi elliptic function expansion method [9] and so on [10].

The main goal of this paper is to find the new and more general exact solutions of the VGKdVmKdV equation by using the generalized Jacobi elliptic functions expansion method [11, 12] proposed recently. The character feature of our method is that, without much extra effort, we can get series of exact solutions using a uniform way. Another advantage of our method is that it also applies to general nonlinear differential equations with nonlinear terms of any order.

This paper is arranged as follows. In Section 2, we briefly describe the generalized Jacobi elliptic function expansion method. In Section 3, several families of solutions to the VGKdV-mKdV equation are obtained. In Section 4, some conclusions are given.

II. SUMMARY OF THE GENERALIZED JACOBI ELLIPTIC FUNCTIONS EXPANSION METHOD

For a given partial differential equation, say, in two variables x and t

$$P(u, u_t, u_x, u_{xt}, u_{tt}, u_{xx}, \dots) = 0 \quad (1)$$

We seek the following formal solutions of the given system:

$$u(\xi) = \sum_{i=0}^N A_i F^i(\xi) + \sum_{i,j=1; i \leq j \leq N} [B_i F^{j-i}(\xi) E^i(\xi) + C_i F^{j-i}(\xi) G^i(\xi) + D_i F^{j-i}(\xi) H^i(\xi)] \quad (2)$$

Where $A_0, A_i, B_i, C_i, D_i, (i=1, 2, \dots, n)$ are time-dependent functions to be determined later. $\xi = \xi(x, t)$ are arbitrary functions with the variables x and t . The parameter N can be determined by balancing the highest order derivative terms with the nonlinear terms in Eq.(2). And $E(\xi), F(\xi), G(\xi), H(\xi)$ are an arbitrary array of the four functions $e = e(\xi), f = f(\xi), g = g(\xi)$ and $h = h(\xi)$, the selection obey the principle which makes the calculation simpler. Here we ansatz

$$\begin{cases} e = \frac{1}{p + qsn\xi + rcn\xi + ldn\xi}, f = \frac{sn\xi}{p + qsn\xi + rcn\xi + ldn\xi} \\ g = \frac{cn\xi}{p + qsn\xi + rcn\xi + ldn\xi}, h = \frac{dn\xi}{p + qsn\xi + rcn\xi + ldn\xi} \end{cases} \quad (3)$$

where p, q, r, l are arbitrary constants, the four function e, f, g, h satisfy the restricted relations(4) and (5a-5d) mentioned in [11, 12].

Substituting (2) along with (4) (5a-5d) into Eq.(1) separately yields four families of polynomial equations for $E(\xi), F(\xi), G(\xi), H(\xi)$. Setting the coefficients

$$\text{of } F^i(\xi) E(\xi)^{j_1} G(\xi)^{j_2} H(\xi)^{j_3} \quad (i=0, 1, 2, \dots)$$

($j_1, \dots, j_3 = 0, 1; j_1 j_2 j_3 = 0$) to zero yields a set of over-determined differential equations (ODEs) in $A_0, A_i, B_i, C_i, D_i, (i=1, 2, \dots, n)$ and $\xi(x, t)$, solving the ODEs by Mathematica and Wu elimination, we can obtain many exact solutions of Eq.(1). according to (2) and (3).

III. EXACT SOLUTIONS OF THE GENERALIZED KDV-MKDV EQUATION

The variable-coefficient generalized KdV-mKdV equation with nonlinear terms of any order

$$u_t + \gamma(t)u_x + \alpha(t)u^n u_x + M(t)u^{2n}u_x + \beta(t)u_{xxx} = R(t) \quad (4)$$

where n is a positive constant, in particular, n can be an arbitrary positive integer, $u = u(x, t)$, $\gamma(t)$, $\alpha(t)$, $M(t)$, $\beta(t)$ and the forced term $R(t)$ are arbitrary functions of x, t . Eq. (6) includes a class of physically important equations.

In fact, if one takes $\gamma(t) = M(t) = 0, n = 1$, Eq. (4) represents VKdV equation [13-15]

$$u_t + \alpha(t)uu_x + \beta(t)u_{xxx} = R(t) \quad (5)$$

If one takes $\gamma(t) = 0, M(t) \neq 0, n = 1$, Eq. (4) becomes VCKdV equation [16-17]

$$u_t + \alpha(t)uu_x + M(t)u^2u_x + \beta(t)u_{xxx} = R(t) \quad (6)$$

If one takes $\gamma(t) = M(t) = R(t) = 0, \beta(t) = 1$, Eq. (4) turns to GVKdV equation [18-19]

$$u_t + \alpha(t)u^n u_x + u_{xxx} = 0 \quad (7)$$

If one takes $R(t) = 0, \alpha(t), \gamma(t), M(t), \beta(t)$ as arbitrary constants, Eq. (4) turns to GKdV-mKdV equation [20-24]

$$u_t + (\gamma + \alpha u^n + M u^{2n})u_x + \beta u_{xxx} = 0 \quad (8)$$

In the follows, we construct exact solutions of Eq. (4).

Using transformation $u = v^n$ yield

$$v^2 v_t + \gamma v^2 v_x + \alpha v^3 v_x + M v^4 v_x + \beta \left[\frac{(1-n)(1-2n)}{n^2} (v_x)^3 + \frac{3(1-n)}{n} v v_x v_{xx} + v^2 v_{xxx} \right] = n v^{1-\frac{1}{n}} R(t) \quad (9)$$

Where

$$u = u(x, t), v = v(x, t), \gamma = \gamma(t), \alpha = \alpha(t), \\ M = M(t), \beta = \beta(t).$$

Making the gauge transformation

$$\xi = k(t)x + \omega(t) \quad (10)$$

We have

$$(\omega_t + k_t x) v^2 v_\xi + k \gamma v^2 v_\xi + k \alpha v^3 v_\xi + k M v^4 v_\xi \\ + k^3 \beta \left[\frac{(1-n)(1-2n)}{n^2} (v_\xi)^3 + \frac{3(1-n)}{n} v v_\xi v_{\xi\xi} + v^2 v_{\xi\xi\xi} \right] = n v^{1-\frac{1}{n}} R(t) \quad (11)$$

where

$$k = k(t), \omega = \omega(t), \omega_t = \frac{d\omega}{dt}, k_t = \frac{dk}{dt}, v_\xi = \frac{dv}{d\xi},$$

$v_{\xi\xi} = \frac{d^{(2)}v}{d\xi^2}, v_{\xi\xi\xi} = \frac{d^{(3)}v}{d\xi^3}, k(t), \omega(t)$ are functions of t to be determined later.

A. The Jacobi Elliptic Function Solutions to Eq. (5)

By balancing the highest-order linear term u''' and the nonlinear uu' in (5), we obtain $N = 2$, thus we assume that (5) have the following solutions:

$$u = c_0 + c_1 e + c_2 f + c_3 g + c_4 h + d_1 e^2 + d_2 f^2 + d_3 g^2 + d_4 h^2 + d_5 fg + d_6 fh + d_7 gh + d_8 ef + d_9 eg + d_{10} eh \quad (12)$$

Where

$u = u(\xi), e = e(\xi), f = f(\xi), g = g(\xi), h = h(\xi), c_i = c_i(t), d_j = d_j(t) (i = 0, \dots, 4; j = 1, \dots, 10)$ and e, f, g, h satisfy (4) and (5a-5d). Substituting (4) and (5a-5d) separately along with (12) into (5) and setting the coefficients of $F^i(\xi)E^{j_1}(\xi)G^{j_2}(\xi)H^{j_3}(\xi) (i=0, 1, 2, \dots) (j_{1..4}=0, 1, j_1 j_2 j_3 j_4=0)$ to zero yield an ODEs with respect to the unknowns $c_i (i = 0, \dots, 4), d_j (j = 1, \dots, 10), \omega, k, p, q, r, l, m$. After solving the ODEs by Mathematica and Wu elimination we could determine the following solutions:

Case 1

$$r = l = 1, q = \pm 1, \beta(t) = C\alpha(t), c_0 = \int R(t)dt,$$

$$d_3 = -\frac{3Ck^2(m^2-2)^2}{m^2-1}, d_4 = \frac{3Ck^2m^4}{1-m^2},$$

$$c_3 = -6k^2C(m^2-2), c_4 = 6Ck^2m^2,$$

$$\omega(t) = \int [-k\alpha(t) \int R(t)dt - k^3C(2m^2+5)\alpha(t)]dt$$

Case 2

$$\beta(t) = C\alpha(t), c_0 = \int R(t)dt,$$

$$d_3 = -\frac{3k^2p^2m^4C}{(m^2-1)}, l = \pm \frac{p}{\sqrt{1-m^2}},$$

$$\omega(t) = \int [-k\alpha(t) \int R(t)dt - 2Ck^3(m^2-2)\alpha(t)]dt$$

Case 3

$$p^2 = 1, q^2 = 1, r = \pm 1, \beta(t) = C\alpha(t),$$

$$c_3 = \pm 12k^2C, d_3 = -12k^2C, c_0 = \int R(t)dt,$$

$$\omega(t) = \int [-k\alpha(t) \int R(t)dt + Ck^3(4m^2-5)\alpha(t)]dt$$

Case 4

$$q = r = 0, \beta(t) = C\alpha(t),$$

$$d_3 = 12Ck^2m^2p^2, c_0 = \int R(t)dt,$$

$$\omega(t) = \int [-k\alpha(t) \int R(t)dt + 4Ck^3(1-2m^2)\alpha(t)]dt$$

where $C \neq 0, k \neq 0$ are arbitrary constants in Case 1-Case 4. The other coefficients $c_i, d_j (i=1, \dots, 4; j=1, \dots, 10)$ don't mention in all above cases are zero. Therefore from (3), (10), (12) and Cases 1-4, we obtain the Jacobi elliptic wave-like solutions to Eq. (5):

$$u_{1,1}(\xi_1) = \int R(t)dt + \frac{-6k^2C(m^2-2)cn\xi_1 + 6Ck^2m^2dn\xi_1}{\pm sn\xi_1 + cn\xi_1 + dn\xi_1} + \frac{3Ck^2(m^2-2)^2cn^2\xi_1 + 3Ck^2m^4dn^2\xi_1}{(1-m^2)(\pm sn\xi_1 + cn\xi_1 + dn\xi_1)^2}$$

$$\xi_1 = kx + \int [-k\alpha(t) \int R(t)dt - k^3C(2m^2+5)\alpha(t)]dt$$

$$u_{1,2}(\xi_2) = \int R(t)dt - \frac{3k^2m^4Ccn^2\xi_2}{(m^2-1)(\sqrt{1-m^2} \pm dn\xi_2)^2}$$

$$\xi_2 = kx + \int [-k\alpha(t) \int R(t)dt - 2Ck^3(m^2-2)\alpha(t)]dt$$

$$u_{1,3}(\xi_3) = \int R(t)dt \pm \frac{12k^2Ccn\xi_3}{1 \pm sn\xi_3 + \varepsilon cn\xi_3} - \frac{12k^2Ccn^2\xi_3}{(1 \pm sn\xi_3 + \varepsilon cn\xi_3)^2}$$

$$\xi_3 = kx + \int [-k\alpha(t) \int R(t)dt + Ck^3(4m^2-5)\alpha(t)]dt$$

$$u_{1,4}(\xi_4) = \int R(t)dt + 12Ck^2m^2cn^2\xi_4$$

$$\xi_4 = kx + \int [-k\alpha(t) \int R(t)dt + 4Ck^3(1-2m^2)\alpha(t)]dt$$

Remark1: If we let $\int R(t)dt = \text{const}, k \rightarrow p, C \rightarrow 1, m \rightarrow k$ or 1, $u_{1,4}$ is equivalent to the Solutions (23, 25, 26) and the famous bell-shape soliton Solution (48) given in [18]. If we let $\int R(t)dt = 0, k \rightarrow p, m \rightarrow k, C \rightarrow 1$, $u_{1,2}$ is equivalent to the Solution (47) given in [18].

Solutions $u_{1,i}(\xi_i) (i=1, 3, 4)$ are degenerated to soliton-like solutions when the modulus $m \rightarrow 1$, and solutions $u_{1,i}(\xi_i) (i=1, 3)$ are degenerated to trigonometric functions solutions when the modulus $m \rightarrow 0$.

The typical structure of new Jacobi elliptic wave-like solution $u_{1,2}, u_{1,4}$ is shown in Fig.1 and Fig.2 with $R(t)=0$. Here $u_{1,4}$ provides us a Jacobi cnoidal wave solution and the famous bell-shaped soliton solution when $m \rightarrow 1$ which covers the large majority of physically interesting solitary waves.

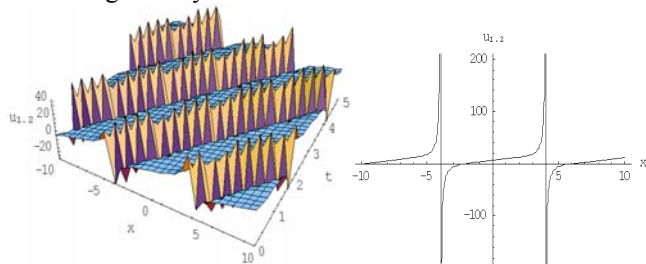


Fig. 1 Simulation of $u_{1,2}$

when $\alpha(t) = k = C = 1, \varepsilon = -1, m = 0.64$ and $t = 0$.

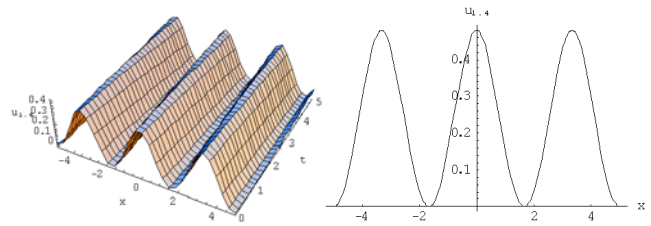


Fig. 2 Simulation of $u_{1,4}$

when $\alpha(t) = 0.025, k = C = 1, \varepsilon = -1, m = 0.2$ and $t = 0$.

B. The Jacobi Elliptic Function Solutions to Eq. (6)

By balancing the highest-order linear term u''' and the nonlinear u^2u' in (6), we obtain $N = 1$, thus we assume that (6) have the following solutions:

$$u = c_0(t) + c_1(t)e + c_2(t)f + c_3(t)g + c_4(t)h \quad (13)$$

with the similar process, we obtain the following solutions:

Case 1

$$p = q, q^2 + r^2 \neq 0, \beta(t) = 2CM(t), k(t) = k,$$

$$c_0(t) = \int R(t)dt, c_1(t) = c_2(t) = c_4(t) = 0,$$

$$c_3(t) = \pm \frac{k}{p} \sqrt{3C((p^2 - r^2)^2(m^2 - 1) - 4p^2r^2)},$$

$$\omega(t) = \int \left[\frac{k\alpha(t)^2}{4M(t)} - k^3CM(t) \right] dt,$$

$$\int R(t)dt = -\frac{\alpha(t)}{2M(t)} \pm \frac{kr((p^2 + r^2) + m^2(p^2 - r^2))}{p} \times \sqrt{\frac{3C}{(m^2(p^2 - r^2)^2 - (p^2 + r^2)^2)}}$$

Case 2

$$p = 1, q = 0, r \neq 0, \beta(t) = 2CM(t), k(t) = k,$$

$$c_0(t) = \int R(t)dt, c_1(t) = c_2(t) = c_4(t) = 0,$$

$$c_3(t) = \pm k \sqrt{12C(r^2 - 1)(m^2(r^2 - 1) - r^2)},$$

$$\omega(t) = \int \left[\frac{k\alpha(t)^2}{4M(t)} - \right.$$

$$\left. \frac{k^3CM(t)(r^2(1+2r^2) + 2m^2(r^2-1)(1-3r^2+2m^2(r^2-1)))}{(1-r^2)(r^2+m^2(1-r^2))} \right] dt$$

$$\int R(t)dt = -\frac{\alpha(t)}{2M(t)}$$

$$\mp kr(1-2r^2+2m^2(r^2-1)) \sqrt{\frac{3C}{(1-r^2)(r^2+m^2(1-r^2))}}$$

where $C \neq 0, k \neq 0$ are arbitrary constants in Case 1-Case 2. Therefore from (3), (10), (13) and Cases 1-2, we obtain the Jacobi elliptic wave-like solutions to Eq. (6):

$$u_{2,1}(\xi_1) = \int R(t)dt \pm \frac{k\sqrt{3C((p^2-r^2)^2(m^2-1)-4p^2r^2)}cn\xi_1}{p^2+p^2sn\xi_1+prcn\xi_1}$$

$$\xi_1 = kx + \int \left[\frac{k\alpha(t)^2}{4M(t)} - k^3CM(t) \right] dt$$

$$u_{2,2}(\xi_2) = \int R(t)dt \pm \frac{k\sqrt{12C(r^2-1)(m^2(r^2-1)-r^2)}cn\xi_2}{1+rcn\xi_2}$$

$$\xi_2 = kx + \int \left[\frac{k\alpha(t)^2}{4M(t)} - \frac{k^3CM(t)(r^2(1+2r^2)+2m^2(r^2-1)(1-3r^2+2m^2(r^2-1)))}{(1-r^2)(r^2+m^2(1-r^2))} \right] dt$$

Remark 2: If we let $m \rightarrow 1$, $u_{2,2}$ contain the Solutions (8,9) in Ref.[25].

Solutions $u_{2,i}(\xi_i)(i=1,2)$ are degenerated to soliton-like solutions when the modulus $m \rightarrow 1$, and solutions $u_{2,i}(\xi_i)(i=1,2)$ are degenerated to trigonometric functions solutions when the modulus $m \rightarrow 0$.

C. Soliton-Like and Trigonometric Function Solutions of Eq. (4) $n \neq 1, R(t) = 0$

By balancing the highest-order linear term v^4v_ξ and the nonlinear $v^2v_{\xi\xi\xi}$ in (11), we obtain $N=1$, thus we assume that (11) have the following solutions:

$$v = c_0(t) + c_1(t)e + c_2(t)f + c_3(t)g + c_4(t)h \quad (14)$$

with the similar process, we obtain the following solutions:

Case 1

$$M(t)\beta(t) = C_2\alpha^2(t), \beta(t) = C_1\alpha(t),$$

$$p = \pm \sqrt{1+q^2 + \frac{k^2C_2(1+n)(2+n)^2}{(1+2n)n^2}},$$

$$c_0(t) = c_1(t) = c_2(t) = c_4(t) = 0, m = 1,$$

$$c_3(t) = \frac{k^2C_1(1+n)(2+n)}{n^2}, l = 0, r = 1,$$

$$\omega(t) = -\int \left(\frac{k^3C_1\alpha(t)}{n^2} + k\gamma(t) \right) dt$$

Case 2

$$M(t)\beta(t) = C_2\alpha^2(t), \beta(t) = C_1\alpha(t),$$

$$r = \pm \sqrt{1-q^2 - \frac{k^2C_2(1+n)(2+n)^2}{(1+2n)n^2}},$$

$$c_0(t) = c_2(t) = c_3(t) = c_4(t) = 0, m = 0,$$

$$c_1(t) = -\frac{k^2C_1(1+n)(2+n)}{n^2}, l = 0, p = 1,$$

$$\omega(t) = -\int \left(\frac{k^3C_1\alpha(t)}{n^2} + k\gamma(t) \right) dt$$

Case 3

$$m = 1, p^2 \neq 1, q = 1, r = l = 0, c_2(t) = c_3(t) = c_4(t) = 0,$$

$$\beta(t) = C_1\alpha(t), M(t) = -\frac{n^2(1+2n)}{4k^2C_1(1+n)(2+n)^2}\alpha(t)$$

$$c_0(t) = \frac{2k^2C_1(1 \pm p)(1+n)(2+n)}{n^2},$$

$$c_1(t) = \pm \frac{2k^2C_1(1+n)(2+n)(p^2-1)}{n^2},$$

$$\omega(t) = -\int \left(\frac{4k^3C_1\alpha(t)}{n^2} + k\gamma(t) \right) dt,$$

where $C_1 \neq 0, C_2, k \neq 0$ are arbitrary constants in Case 1-Case 3. Therefore from (3), (10), (14) and Cases 1-3, noticed $u = v^n$, we obtain the following soliton-like and trigonometric function solutions for Eq. (6) are expressed by

$$u_{3,1}(x,t) = \left\{ \frac{B_{3,1}}{A_{3,1}} \right\}^{\frac{1}{n}},$$

$$B_{3,1} = \frac{k^2C_1(1+n)(2+n)}{n^2} \times$$

$$\sec h \left[kx - \int \left(\frac{k^3C_1\alpha(t)}{n^2} + k\gamma(t) \right) dt \right]$$

$$A_{3,1} = \pm \sqrt{1+q^2 + \frac{k^2C_2(1+n)(2+n)^2}{(1+2n)n^2}} +$$

$$q \tanh \left[kx - \int \left(\frac{k^3C_1\alpha(t)}{n^2} + k\gamma(t) \right) dt \right]$$

$$+ \sec h \left[kx - \int \left(\frac{k^3C_1\alpha(t)}{n^2} + k\gamma(t) \right) dt \right]$$

$$u_{3,2}(x,t) = \left\{ \frac{B_{3,2}}{A_{3,2}} \right\}^{\frac{1}{n}},$$

$$B_{3,2} = -\frac{k^2 C_1 (1+n)(2+n)}{n^2} \times$$

$$\sec[kx - \int (\frac{k^3 C_1 \alpha(t)}{n^2} + k\gamma(t)) dt]$$

$$A_{3,2} = \pm \sqrt{1 - q^2 - \frac{k^2 C_2 (1+n)(2+n)^2}{(1+2n)n^2}}$$

$$+ q \tan[kx - \int (\frac{k^3 C_1 \alpha(t)}{n^2} + k\gamma(t)) dt]$$

$$+ \sec[kx - \int (\frac{k^3 C_1 \alpha(t)}{n^2} + k\gamma(t)) dt]$$

$$u_{3,3}(x,t) = \left\{ \frac{2k^2 C_1 (1 \pm p)(1+n)(2+n)}{n^2} + \frac{B_{3,3}}{A_{3,3}} \right\}^{\frac{1}{n}}$$

$$B_{3,3} = \pm \frac{2k^2 C_1 (1+n)(2+n)(p^2 - 1)}{n^2}$$

$$A_{3,3} = p + \tanh[kx - \int (\frac{4k^3 C_1 \alpha(t)}{n^2} + k\gamma(t)) dt]$$

If we let $q = q_1 = 0$ or

$$q = q_2 = \pm \sqrt{-1 - \frac{k^2 C_2 (1+n)(2+n)^2}{(1+2n)n^2}} \text{ and}$$

$$n \rightarrow p, C_2 \rightarrow \frac{\gamma}{\beta^2}, k \rightarrow p\sqrt{A}, \gamma(t) \rightarrow \alpha,$$

$$C_1 \rightarrow \frac{1}{\beta}, \alpha(t) \rightarrow \beta \text{ in } u_{3,1}(x,t),$$

we have

$$u_{3,1}(x,t) \rightarrow u_{3,1.1}(x,t) =$$

$$\left\{ \frac{\pm A(1+p)(2+p) \operatorname{sech}[p\sqrt{A}(x - (\alpha + A)t + \xi_0)]}{\sqrt{\beta^2 + A\gamma \frac{(1+p)(2+p)^2}{1+2p} \pm \beta \operatorname{sech}[p\sqrt{A}(x - (\alpha + A)t + \xi_0)]}} \right\}^{\frac{1}{p}}$$

$$u_{3,1}(x,t) \rightarrow u_{3,1.2} =$$

$$\left\{ \frac{\pm A(1+p)(2+p) \operatorname{csch}[p\sqrt{A}(x - (\alpha + A)t + \xi_0)]}{\sqrt{-\beta^2 - A\gamma \frac{(1+p)(2+p)^2}{1+2p} \pm \beta \operatorname{csch}[p\sqrt{A}(x - (\alpha + A)t + \xi_0)]}} \right\}^{\frac{1}{p}}$$

$u_{3,1.1}(x,t), u_{3,1.2}(x,t)$ is just the solution (28) (29) in Ref.[20]

Remark 3: If we let

$$n \rightarrow p, C_2 \rightarrow \frac{\gamma}{\beta^2}, k \rightarrow p\sqrt{-A}, \gamma(t) \rightarrow \alpha,$$

$$C_1 \rightarrow \frac{1}{\beta}, \alpha(t) \rightarrow \beta, q = 0$$

or $q = \pm \sqrt{1 - \frac{k^2 C_2 (1+n)(2+n)^2}{(1+2n)n^2}}$, $u_{3,2}(x,t)$ turns to the Solutions (32)(33) in Ref.[20]. Noticed that $\tanh \xi = \frac{\sinh \xi}{\cosh \xi}$, $u_{3,3}(x,t)$ contains the Solution (3.18) in Ref.[22]. Our method contains all the results mentioned by the G'/G method [22], the improved sub-ODE method [20], [21] and auxiliary equation technique [18], etc.

The properties of the new soliton-like wave solutions $u_{3,1}$ and periodic-like solutions $u_{3,2}$ are shown in Fig. 3 and Fig. 4.

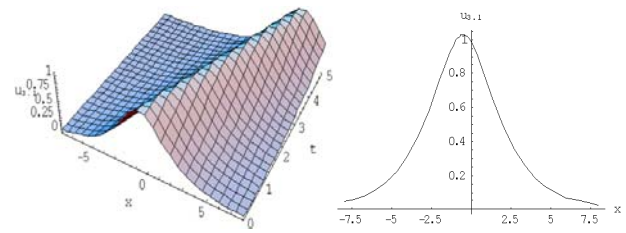


Fig. 3 Solution $u_{3,1}$

when $n = 2, k = q = \alpha(t) = \gamma(t) = 1, C_1 = C_2 = 1$ and $t = 0$.

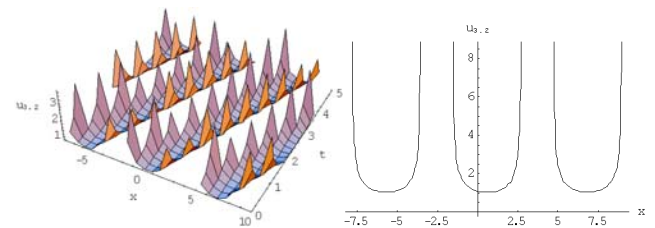


Fig. 4 Solution $u_{3,2}$

when $n = 2, k = q = \alpha(t) = \gamma(t) = 1, C_1 = C_2 = 1$ and $t = 0$.

Remark 4: All the solutions obtained in this paper for Eq. (4) have been checked by Mathematica software.

To our knowledge, the explicit solutions except $(u_{1,2}, u_{1,4}, u_{2,2})$ we obtained here to Eq. (4) are not shown in the previous literature. They are new exact solutions of Eq. (4).

IV. CONCLUSIONS

In this paper, we have found abundant new types of exact solutions for the variable-coefficient generalized KdV-mKdV equation by using the generalized Jacobi elliptic functions expansion method and computerized symbolic computation. More importantly, our method is quite simple and powerful to find new solutions to various kinds of nonlinear evolution equations, such as Schrodinger equation, Boussinesq equation etc. We believe that this method should play an important role for finding exact solutions in the mathematical physics.

ACKNOWLEDGMENTS

The work was supported by the National Nature Science Foundation of China (Grant No.61070231), the Outstanding Personnel Program in Six Fields of Jiangsu (Grant No.2009188) and the Scientific Research Foundation of Nanjing Institute of Technology (Grant No. QKJA2010011).

REFERENCES

- [1] M.J. Ablowitz, P.A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press, New York, 1991.
- [2] X.B.Hu, W.X.Ma, Application of Hirota's bilinear formalism to the Toeplitz lattice some special soliton-like solutions, *Phys. Lett. A*, 293 (2002) 161-165.
- [3] E.G.Fan, Two new applications of the homogeneous balance method, *Phys. Lett. A*, 265 (2000) 353-358.
- [4] D.C.Lu, B.J.Hong, Backlund transformation and n-soliton-like solutions to the combined KdV-Burgers equation with variable coefficients, *Int.J.Nonlinear.Sci*, 1(2) (2006)3-10.
- [5] Matveev.V.A, Salle.M.A, *Darboux transformations and solitons*, Berlin, Heidelberg: Springer-Verlag. (1991).
- [6] E.G.Fan, Extended tanh-function method and its applications to nonlinear equations, *Phys Lett A*, 277 (2000)212-220.
- [7] Y.B.Zhou, M.L.Wang, Y.M.Wang, Periodic wave solutions to a coupled KdV equations with variable coefficients, *Phys.Lett.A*,308(2003)31-37.
- [8] D.C.Lu, B.J.Hong, New exact solutions for the (2+1)-dimensional Generalized Broer-Kaup System, *Appl.Math. Comput*,199(2) (2008) 572-580.
- [9] W.H.Huang, Y.L.Liu, Jacobi elliptic function solutions of the Ablowitz-Ladik discrete nonlinear Schrödinger system, *Chaos, Solitons and Fractals*,40 (2009) 786-792.
- [10] B.J.Hong, New Jacobi elliptic functions solutions for the variable-coefficient mKdV equation, *Appl. Math. Comput*,215(8)(2009)2908-2913.
- [11] B.J.Hong, D.C.Lu, New Jacobi elliptic function-like solutions for the general KdV equation with variable coefficients, *Math.Compu.Modelling*,55 (2012) 1594-1600.
- [12] B.J.Hong, New exact Jacobi elliptic functions solutions for the generalized coupled Hirota-Satsuma KdV system, *Appl.Math.Comput*, 217 (2010) 472-479.
- [13] R. Sabry, M.A. Zahran, Engui Fan, A new generalized expansion method and its application in finding explicit exact solutions for a generalized variable coefficients KdV equation, *Phys.Lett.A*,326 (2004) 93-101.
- [14] M.L.Wang, Y.M.Wang, A new Bäcklund transformation and multi-soliton solutions to the KdV equation with general variable coefficients, *Phys.Lett. A*, 287 (2001) 211-216.
- [15] E.G.Fan, Auto-Bäcklund transformation and similarity reductions for general variable coefficient KdV equations, *Phys.Lett.A*, 294 (2002) 26-30.
- [16] D.C. Lu, B.J. Hong, L.X. Tian, Explicit and exact solutions to the variable coefficient combined KdV equation with forced term, *Acta Phys. Sin*, 55 (11)(2006) 5617-5622 (in Chinese).
- [17] Taogetusang, Sirendaorji, The auxiliary equation for constructing the exact solutions of the variable coefficient combined KdV equation with forcible term. *Acta Phys. Sin*, 57 (03)(2008) 1295-1300 (in Chinese).
- [18] Y.Zhang, S.Y. Lai, J.Yin, Y.H.Wu, The application of the auxiliary equation technique to a generalized mKdV equation with variable coefficients, *J. Comput. Appl. Math*,223 (2009) 75-85.
- [19] A.H. Khater, W. Malfliet, D.K. Callebaut, E.S. Kamel, Travelling wave solutions of some classes of nonlinear evolution equations in (1+1) and (2+1) dimensions, *J. Comput. Appl. Math*. 140 (2002) 469-477.
- [20] S. Zhang, W. Wang, J.L. Tong, The improved sub-ODE method for a generalized KdV-mKdV equation with nonlinear terms of any order, *Phys. Lett. A*,372(2008) 3808-3813.
- [21] X.Z. Li, M.L. Wang, A sub-ODE method for finding exact solutions of a generalized KdV-mKdV equation with high-order nonlinear terms, *Phys. Lett. A*,361 (2007) 115-118.
- [22] E.M.E. Zayed, Khaled A. Gepreel, Some applications of the G'/G-expansion method to non-linear partial differential equations, *Appl. Math. Comput*, 212 (2009) 1-13.
- [23] C.M. Khalique, A. Biswas, Optical Solitons with parabolic and dual power law nonlinearity via Lie symmetry analysis, *J. Electromagn. Waves Appl.*, 23(2009) 963-973.
- [24] M. Antonova, A. Biswas, Adiabatic parameter dynamics of perturbed solitary waves, *Commun. Nonlinear Sci. Numer. Simul.*, 14 (2009) 734-748.
- [25] Y.R. Shi, K.P. Lv, W.S.H. Duan, et al., Explicit and exact solutions of the combined KdV equation, *Acta Phys. Sinica*, 52 (02) (2003)0267-0270(in Chinese).