New Traveling Wave Solutions of the Boussinesq Equation Using a New Generalized Mapping Method

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***Abstract*-In this paper, a generalized mapping method for finding the exact traveling wave solutions of a nonlinear partial differential equation is discussed. Firstly, some new solutions of an auxiliary ordinary differential equation are introduced. They are then used to generate new exact solutions for the Boussinesq equation. The new solutions are then grouped into ten families and the properties of each family of solutions are demonstrated. We should also emphasize here that the developed method can also be applied to a large variety of nonlinear partial differential equations in physics and mechanics.**

***Keywords*- Traveling Wave Solutions; Boussinesq Equation; Mapping Method; Solitons**

I. INTRODUCTION

The purpose of this paper is to find new classes of exact traveling wave solutions, by using a generalized mapping method, for the nonlinear Boussinesq equation

$$u_{tt} = u_{xx} + u_{xxxx} + 3(u^2)_{xx} \ . \tag{1.1}$$

The above model was original introduced by Boussinesq to describe the propagation of long waves in shallow water [1], where $u(t, x)$ is the elevation of the free surface of the fluid, and the subscripts denote partial derivatives. The equation also arises in many other physical applications such as nonlinear lattice waves [2], iron sound waves in plasma [3], and vibrations in a nonlinear string [4]. It was also applied to the study of the percolation of water in porous subsurface strata [5].

Ref. [6] and [7] studied the local existence and blow-up, instability and strong instability of solitary-waves, long-time behavior of solutions and nonlinear scattering theory of (1.1). Felipe studied the global existence of small solutions for a generalized Boussinesq equation [8]. In [9], the existence and nonexistence of global weak solutions to the Cauchy problem for the multi-dimensional Boussinesq type equation were studied, and it was proved that the Cauchy problem admits a global weak solution under certain assumptions. The weak solution is regularized and the strong solution is unique when the space dimension $N = 1$. Wang et al. discussed the decay and scattering of solutions for a generalized Boussinesq equation in [10].

Traveling wave solutions represents an important type of solutions for nonlinear partial differential equations as many nonlinear partial differential equations have been found to have a variety of traveling wave solutions. It is well-known that the investigation of the exact solutions of nonlinear partial differential equations (PDEs) plays an important role in the study of nonlinear physical phenomena. Exact traveling wave solutions are useful for verifying the accuracy and stability of popular numerical schemes such as the finite difference and finite element methods. It is thus important to find new exact solutions for nonlinear PDEs.

In the past few decades, several effective search methods for the solutions of nonlinear PDEs have been proposed in the literature. These include the variational iteration method [11-13], the tanh method [14], the extend tanh-function method [15], the exp-function method [16, 17], the F-expansion method [18], the bifurcation method [19], the Jacobian elliptic function method [20], the Weierstrass's elliptic function method [21], the reduction of order methods [1, 22], cosh/sinh ansatz I-III method [5], the mapping method [23-25], and the (G'/G)-expansion method [26]. Most of these methods have advantages on one hand and disadvantages on the other hand. Recently, some composite methods were purposed to solve various nonlinear equations. The mapping method combined with other methods has been paid particular attention.

Eq. (1.1) posses traveling wave solutions called solitary waves and Boussinesq was the first to give a scientific explanation of their existence. In [5], [24] and [27], several traveling wave solutions of the Boussinesq equation were obtained by using the mapping method or ansatze methods (cosh-sinh ansatze method, sine-cosine ansatze methods and tanh ansatze method). Various traveling wave solutions have also be obtained for the generalized Boussinesq equations and high dimensional Boussinesq equation [1, 4].

In this paper, we extend the mapping method developed in [4, 23-25], and deduce many new classes of exact traveling wave solutions for the Boussinesq equation, which are completely different from the results obtained in [5, 24, 27]. We shall emphasize that most of our new solutions cannot be obtained by other methods, and that the method can be applied to solve many other nonlinear evolution equations. The rest of this paper is organized as follows. In the next section, we introduce an

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auxiliary ordinary differential equation and obtain some new elementary form solutions and relevant results. In Section 3, the key idea of our method is described. In Section 4, the proposed method is applied to the Boussinesq equation. Conclusions are presented in Section 5.

II. PRELIMINARY RESULTS

In order to obtain new elementary function classes of travelling wave solutions, we consider the following auxiliary ordinary differential equation

$$\left(\frac{d\varphi(\xi)}{d\xi}\right)=h_0+h_1\varphi(\xi)+h_2\varphi(\xi)^2+h_3\varphi(\xi)^3+h_4\varphi(\xi)^4, \tag{2.1}$$

where h_i, i = 0, ..., 4, are constants. Various solutions of the ODE (2.1) have been constructed by using the Jacobian elliptic functions and etc. The results were exploited in generating Jacobi function class of traveling wave solutions of nonlinear partial differential equations [20, 23, 24]. Obviously, different types of exact solutions of the auxiliary Equation (2.1) may result in different types of exact traveling wave solutions for nonlinear partial differential equations. Hence, we seek new elementary function solutions of the ODE (2.1) in order to find new elementary function classes of traveling wave solutions for Eq. (1.1). The capability and power of computer algebra software such as Maple or Mathematica have increased dramatically over the past decade. Hence, a direct search for exact solutions for the ODE (2.1) is now much more viable. In this paper, the parameters a, b, c and d in different equations may be different in this paper. Using symbolic calculations via Maple, we obtain the following results.

Theorem 1. Let c, d be arbitrary constants, $\beta,\delta\in\{-1,1\}$.

(1) If h_0=h_1 = 0, Eq. (2.1) possesses the following solutions:

$$\varphi_1=\frac{\delta}{\sqrt{h_4}\xi-c},h_2=h_3=0,h_4\neq 0, \tag{2.2}$$

$$\varphi_2=\frac{4h_3}{h_3^2(\xi-c)^2-4h_4},h_2=0,h_3\neq 0 \tag{2.3}$$

$$\varphi_3=\frac{4h_2\delta e^{\sqrt{h_2}(\beta\xi-c)}}{(h_3e^{\sqrt{h_2}(\beta\xi-c)}+1)^2},h_2\neq 0, \tag{2.4}$$

$$\varphi_4=\frac{4h_2\delta e^{\sqrt{h_2}(\beta\xi-c)}}{(e^{\sqrt{h_2}(\beta\xi-c)}-h_3)^2-4h_2h_4},h_2\neq 0, \tag{2.5}$$

$$\varphi_5=\frac{h_2\delta}{h_3}\coth(\frac{\sqrt{h_2}}{2}(\beta\xi-c))-\frac{h_2}{h_3},h_2\neq 0,h_3^2=4h_2h_4, \tag{2.6}$$

$$\varphi_6=\frac{h_2\delta}{h_3}\tanh(\frac{\sqrt{h_2}}{2}(\beta\xi-c))-\frac{h_2}{h_3},h_2h_3\neq 0,h_3^2=4h_2h_4, \tag{2.7}$$

$$\varphi_7=\frac{h_2\sec h^2(\frac{\sqrt{h_2}}{2}(\beta\xi-c))}{2\sqrt{h_2h_4}\tanh(\frac{\sqrt{h_2}}{2}(\beta\xi-c))-h_3},h_2\neq 0, \tag{2.8}$$

$$\varphi_8=\frac{-h_2}{h_3\cos^2(\frac{\sqrt{-h_2}}{2}(\xi-c))+\delta\sqrt{-h_2h_4}\sin(\sqrt{-h_2}(\xi-c))},h_2h_4\neq 0, \tag{2.9}$$

$$\varphi_9=\frac{-h_2}{h_3\sin^2(\frac{\sqrt{-h_2}}{2}(\xi-c))+\delta\sqrt{-h_2h_4}\sin(\sqrt{-h_2}(\xi-c))},h_2h_4\neq 0, \tag{2.10}$$

$$\varphi_{10}=\frac{h_2}{h_3\sinh^2(\frac{\sqrt{h_2}}{2}(\beta\xi-c))+\delta\sqrt{h_2h_4}\sinh(\sqrt{h_2}(\beta\xi-c))},h_2h_4\neq 0, \tag{2.11}$$

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$$\varphi_{11}=\frac{-h_2}{h_3\cosh^2(\frac{\sqrt{h_2}}{2}(\beta\xi-c))+\delta\sqrt{h_2h_4}\sinh(\sqrt{h_2}(\beta\xi-c))},h_2h_4\neq 0, \tag{2.12}$$

$$\varphi_{12}=\frac{2h_2}{-h_3+\delta\sqrt{h_3^2-4h_2h_4}\sin(\sqrt{-h_2}(\xi-c))},\quad h_2(h_3^2-4h_2h_4)\neq 0, \tag{2.13}$$

$$\varphi_{13}=\frac{2h_2}{-h_3+\delta\sqrt{4h_2h_4-h_3^2}\sinh(\sqrt{h_2}(\beta\xi-c))},\quad h_2(h_3^2-4h_2h_4)\neq 0, \tag{2.14}$$

$$\varphi_{14}=\frac{2h_2}{-h_3+\delta\sqrt{h_3^2-4h_2h_4}\cosh(\sqrt{h_2}(\beta\xi-c))},\quad h_2(h_3^2-4h_2h_4)\neq 0, \tag{2.15}$$

(2) If $h_1=h_3=0$, Eq. (2.1) possesses the following solutions:

$$\varphi_{15}=\delta\sqrt{\frac{h_2}{2h_4}}\tan(\sqrt{\frac{h_2}{2}}(\xi-c)),\quad h_2h_4\neq 0,\ h_2^2=4h_0h_4, \tag{2.16}$$

$$\varphi_{16}=\delta\sqrt{-\frac{h_2}{2h_4}}\tanh(\sqrt{-\frac{h_2}{2}}(\beta\xi-c)),\ h_2h_4\neq 0, h_2^2=4h_0h_4, \tag{2.17}$$

(3) If $h_3=h_4=0$, Eq. (2.1) possesses the following solutions:

$$\varphi_{17}=\delta\sqrt{h_0}(\xi-c),\qquad h_1=h_2=0, \tag{2.18}$$

$$\varphi_{18}=\frac{h_1}{4}(\xi-c)^2-\frac{h_0}{h_1},\quad h_1\neq 0, h_2=0, \tag{2.19}$$

$$\varphi_{19}=de^{\sqrt{h_2}(\beta\xi-c)}-\frac{h_1}{2h_2},\quad h_2\neq 0, h_1^2=4h_0h_2, \tag{2.20}$$

$$\varphi_{20}=\frac{\delta}{2\sqrt{h_2}}e^{\sqrt{h_2}(\beta\xi-c)}+\frac{\delta(h_1^2-4h_0h_2)}{8h_2\sqrt{h_2}}e^{-\sqrt{h_2}(\beta\xi-c)}-\frac{h_1}{2h_2}, h_2\neq 0, \tag{2.21}$$

$$\varphi_{21}=\frac{\delta\sqrt{h_1^2-4h_0h_2}}{2h_2}\sin(\sqrt{-h_2}(\xi-c))-\frac{h_1}{2h_2}, h_2(h_1^2-4h_0h_2)\neq 0, \tag{2.22}$$

$$\varphi_{22}=\frac{\delta\sqrt{h_1^2-4h_0h_2}}{2h_2}\sinh(\sqrt{h_2}(\xi-c))-\frac{h_1}{2h_2},\quad h_2(h_1^2-4h_0h_2)\neq 0, \tag{2.23}$$

Remark 1. When $h_0=h_1=h_3=0$, $\varphi_8=\varphi_9$, and $\varphi_{10}=\varphi_{11}$. Under this case, φ_8 (or φ_9) and φ_{12} are linearly dependent, φ_{10} (or φ_{11}) and φ_{13} are also linearly dependent. When $h_0=h_1=h_3=h_4=0$, φ_3 and φ_{19} are linearly dependent, except for the above cases, the solutions are linearly independent.

All of the above results are new solutions which were not reported before, and include some existing results as special cases. For example, when $c=0$, $h_4=0$, φ_2 becomes Eq. (2.35) in [23]; when $\beta=1$, $c=0$, φ_7 becomes Eq. (7) of Table 1 in [25]; when $c=0$, $h_0=0$, $\delta=1$, φ_{21} becomes Eq. (2.10) in [23], and φ_{22} becomes Eq.(2.11) in [23], etc.

Remark 2. Most of the solutions of the ODE (2.1) reported before are in terms of Jacobi elliptic functions or Weierstrass Functions (see [20, 24] and therein), while the above solutions are in terms of elementary functions. With elementary functions, it is easier to derive analytical expressions for the properties of solutions.

Remark 3. Some results in Theorem 1 are to be used to generate new traveling wave solutions for the Boussinesq Equation (1.1) in this paper, the others may be used to construct traveling wave solutions for other nonlinear partial equations.

Theorem 1 can be generalized further. In some cases, the solutions of Eq. (2.1) can be used to generate additional solutions. Through lengthy calculation, we can readily verify the following results. Note that Maple can be used to help us for the calculation.

Theorem 2. Let a, b and c be arbitrary constants, and $a\neq 0$, $\beta=\pm 1$. Suppose that φ is a solution of ODE (2.1) with

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coefficients $h_k = \hat{h}_k$, k=0,…,4, where $\hat{h}_k$ are given constants such that $\hat{h}_1 = \frac{b}{a}\hat{h}_3$, and $\hat{h}_0 = \frac{b^2}{a^2}\hat{h}_4$. Then

$$a\varphi(\beta\xi - c) + \frac{b}{\varphi(\beta\xi - c)}$$

is a solution of ODE(2.1) with coefficients

$$h_0 = -4ab\hat{h}_2 + 8b^2\hat{h}_4, \quad h_1 = -4b\hat{h}_3,$$

$$h_2 = \frac{a\hat{h}_2 - 6b\hat{h}_4}{a}, \quad h_3 = \frac{\hat{h}_3}{a}, \quad h_4 = \frac{\hat{h}_4}{a^2}.$$

Theorem 3. Assume that *a, b, c* and *d* are arbitrary constants. Then $ae^{\xi - c} + be^{-\xi + c} + d$ is a solution of the following equation

$$\left[\frac{d\varphi(\xi)}{d\xi}\right]^2 = d^2 - 4ab - 2d\varphi(\xi) + \varphi(\xi)^2.$$

III. THE METHOD OF SOLUTION

The nonlinear PDE in two independent variables

$$H(u, u_t, u_x, u_{tt}, u_{xx}, u_{tx}, \cdots\cdots) = 0 \tag{3.1}$$

can be transformed to a nonlinear ODE

$$P(\bar{u}, \bar{u}', \bar{u}'', \bar{u}''', \cdots\cdots) = 0 \tag{3.2}$$

by letting $u(t, x) = \bar{u}(\xi)$, where $\xi = k(x - vt)$ in which $k > 0$ is the wave number and v is the traveling wave velocity. Both H and P are polynomials.

We will consider candidate traveling wave solutions that take the form

$$u(t, x) = \bar{u}(\xi) = \sum_{j=-N}^{j=N} c_i \varphi(\xi)^j, \tag{3.3}$$

where N is an integer, φ is a non-trivial solution of the ODE (2.1) with coefficients h_k $k = 0, \cdots, 4$ and c_j, $j = 0, \pm 1, \pm 2, \cdots, \pm N$ are constants with $c_N \neq 0$, or $c_{-N} \neq 0$. In most cases, we will choose N so that the degrees of the highest-order derivative terms and the highest-order nonlinear term in the ODE (3.2) are balanced. However, this does not always result in an integral value for N. In this case, it is sometimes possible to proceed by letting $\bar{u} = w^{1/\gamma}$, where γ is the denominator of the fractional value of N (assuming that the denominator and the numerator have no common factors), and solving the resulting equation for w. Substituting (3.3) into (3.1) along with Eq. (2.1) yields an equation in powers of φ. Then setting all coefficients of φ^i $(i = -N, \cdots, N)$ of the resulting equation to be zero, we obtain an over-determined system of nonlinear algebraic equations of c_i, $i = -N, \cdots, N$. Solving the result system, we get the values of c_i, $i = -N, \cdots, N$. Depending on the form of *H, k* and *v* will be determined or remain as free parameters. Note that Maple can be used to help the calculation.

Having determined these parameters, by using (3.3), we can obtain the exact traveling solutions of the nonlinear Equation (3.1).

IV. NEW EXACT SOLUTIONS OF THE BOUSSINESQ EQUATION

Let $u(t, x) = \bar{u}(\xi)$, where ξ is as defined in Section 3. Equation (1.1) then becomes

$$v^2\bar{u}'' = \bar{u}'' + k^2\bar{u}^{(4)} + 3(\bar{u}^2)''. \tag{4.1}$$

Balancing $(\bar{u}^2)''$ and $\bar{u}^{(4)}$ gives $2N+2 = N+4$, and hence $N = 2$. Thus, we will search for traveling wave solutions of the form

$$\bar{u}(\xi) = c_{-2}\varphi(\xi)^{-2} + c_{-1}\varphi(\xi)^{-1} + c_0 + c_1\varphi(\xi) + c_2\varphi(\xi)^2, \tag{4.2}$$

where $\varphi(\xi)$ is a solution of the ODE (2.1) with coefficients h_k $k = 0, \cdots, 4$ Substituting (4.2) into (4.1) along with Eq.(2.1), and

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then setting all coefficients of φ^i (i=-2,…,2) of the resulting system to be zero, we get the following over-determined system of nonlinear algebraic equations with respect to c_{-2}, c_{-1}, c_1, c_0 and c_2.

$$p6 := 60c_2^2h_4 + 120k^2c_2h_4^2 = 0,$$

$$m6 := 120k^2c_{-2}h_0^2 + 60c_{-2}^2h_0 = 0,$$

$$p5 := 168k^2c_2h_3h_4 + 54c_2^2h_3 + 24k^2c_1h_4^2 + 72c_1c_2h_4 = 0,$$

$$m5 := 54c_{-2}^2h_1 + 168k^2c_{-2}h_0h_1 + 24k^2c_{-1}h_0^2 + 72c_{-2}c_{-1}h_0 = 0,$$

$$\begin{aligned} p4 := {} & 63c_1c_2h_3 - 6c_2h_4v^2 + 18c_1^2h_4 + \frac{105}{2}k^2c_2h_3^2 + 6c_2h_4 + 48c_2^2h_2 \\ & + 36c_0c_2h_4 + 120k^2c_2h_2h_4 + 30k^2c_1h_3h_4 = 0, \end{aligned}$$

$$\begin{aligned} m4 := {} & 30k^2c_{-1}h_0h_1 + 36c_0c_{-2}h_0 + 48c_{-2}^2h_2 + 63c_{-2}c_{-1}h_1 + 120k^2c_{-2}h_2h_0 \\ & + \frac{105}{2}k^2c_{-2}h_1^2 + 6c_{-2}h_0 + 18c_{-1}^2h_0 - 6c_{-2}h_0v^2 = 0, \end{aligned}$$

$$\begin{aligned} p3 := {} & 2c_1h_4 + 5c_2h_3 + 15c_1^2h_3 + 42c_2^2h_1 + 12c_{-1}c_2h_4 - 2c_1h_4v^2 \\ & - 5c_2h_3v^2 + 12c_0c_1h_4 + 30c_0c_2h_3 + 54c_1c_2h_2 + 65k^2c_2h_3 \\ & + 90k^2c_2h_1h_4 + 20k^2c_1h_2h_4 + \frac{15}{2}k^2c_1h_3^2 = 0, \end{aligned}$$

$$\begin{aligned} m3 := {} & 12c_{-2}c_1h_0 + 12c_0c_{-1}h_0 + 42c_{-2}^2h_3 - 2c_{-1}h_0v^2 + 65k^2c_{-2}h_1h_2 \\ & + 15c_{-1}^2h_1 + \frac{15}{2}k^2c_{-1}h_1^2 + 2c_{-1}h_0 + 5c_{-2}h_1 + 54c_{-2}c_{-1}h_2 \\ & + 90k^2c_{-2}h_3h_0 + 20k^2c_{-1}h_2h_0 + 30c_0c_{-2}h_1 - 5c_{-2}h_1v^2 = 0, \end{aligned}$$

$$\begin{aligned} p2 := {} & 36c_2^2h_0 + \frac{3}{2}c_1h_3 + 4c_2h_2 + 12c_1^2h_2 - \frac{3}{2}c_1h_3v^2 + 9c_{-1}c_2h_3 \\ & - 4c_2h_2v^2 + 9c_0c_1h_3 + 24c_0c_2h_2 + 45c_1c_2h_1 + 42k^2c_2h_1h_3 \\ & + 72k^2c_2h_0h_4 + \frac{15}{2}k^2c_1h_2h_3 + 15k^2c_1h_1h_4 + 16k^2c_2h_2^2 = 0, \end{aligned}$$

$$\begin{aligned} m2 := {} & 36c_{-2}^2h_4 + + \frac{15}{2}k^2c_{-1}h_1h_2 + \frac{3}{2}c_{-1}h_1 + 16k^2c_{-2}h_2^2 + 12c_{-1}^2h_2 \\ & + 4c_{-2}h_2 - \frac{3}{2}c_{-1}h_1v^2 + 9c_{-2}c_1h_1 + 45c_{-2}c_{-1}h_3 + 42k^2c_{-2}h_3h_1 \\ & + 72k^2c_{-2}h_4h_0 + 15k^2c_{-1}h_3h_0 + 24c_0c_{-2}h_2 - 4c_{-2}h_2v^2 + 9c_0c_{-1}h_1 = 0, \end{aligned}$$

$$\begin{aligned} p1 := {} & 3c_2h_1 + 9c_1^2h_1 - c_1h_2v^2 + c_1h_2 + 36c_1c_2h_0 + 6c_{-1}c_2h_2 \\ & - 3c_2h_1v^2 + 6c_0c_1h_2 + 18c_0c_2h_1 + 30k^2c_2h_0h_3 + 15k^2c_2h_1h_2 \\ & + k^2c_1h_2^2 + \frac{9}{2}k^2c_1h_1h_3 + 12k^2c_1h_0h_4 = 0, \end{aligned}$$

$$\begin{aligned} m1 := {} & c_{-1}h_2 + 9c_{-1}^2h_3 - c_{-1}h_2v^2 + 6c_{-2}c_1h_2 + 36c_{-2}c_{-1}h_4 + 15k^2c_{-2}h_2h_3 \\ & + 30k^2c_{-2}h_4h_1 + \frac{9}{2}k^2c_{-1}h_3h_1 + 12k^2c_{-1}h_4h_0 + k^2c_{-1}h_2^2 \\ & + 18c_0c_{-2}h_3 - 3c_{-2}h_3v^2 + 3c_{-2}h_3 + 6c_0c_{-1}h_2 = 0, \end{aligned}$$

$$\begin{aligned} z0 := {} & 2c_2h_0 + 6c_1^2h_0 + \frac{1}{2}c_1h_1 + 3c_0c_{-1}h_3 - \frac{1}{2}c_1h_1v^2 + \frac{1}{2}c_{-1}h_3 \\ & + 12c_0c_2h_0 + 3c_{-1}c_2h_1 - 2c_2h_0v^2 + 3c_0c_1h_1 + \frac{1}{2}k^2c_{-1}h_2h_3 \\ & + 3k^2c_{-1}h_4h_1 + 8k^2c_2h_2h_0 - \frac{1}{2}c_{-1}h_3v^2 + 3k^2c_1h_0h_3 \\ & + \frac{1}{2}k^2c_1h_1h_2 + 2c_{-2}h_4 + 3c_{-2}c_1h_3 + 8k^2c_{-2}h_2h_4 + \frac{3}{2}k^2c_{-2}h_3^2 \\ & + 12c_0c_{-2}h_4 + 6c_{-1}^2h_4 - 2c_{-2}h_4v^2 + \frac{3}{2}k^2c_2h_1^2 = 0, \end{aligned}$$

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For $h_4 \neq 0$, $h_0 = (16h_2^2h_4^2 - 8h_2h_3^2h_4 + h_3^4)/64h_4^3$, and $h_1 = h_3(h_3^2 - 4h_2h_4)/(8h_4^2)$, by solving the above over-determined system with the help of the Maple software, we obtain the following sufficient conditions for $\bar{u}$ to satisfy (1.1):

$$\begin{cases} c_0 = \dfrac{4h_4v^2 + 3h_3^2k^2 - 4h_4 - 16h_2h_4k^2}{24h_4}, \\ c_1 = -h_3k^2, \\ c_2 = -2h_4k^2, \\ c_{-1} = -h_1k^2, \\ c_{-2} = -2h_0k^2. \end{cases}$$

That is, if a solution φ of the ODE (2.1) with coefficients satisfying $h_4 \neq 0$, $h_0 = (16h_2^2h_4^2 - 8h_2h_3^2h_4 + h_3^4)/(64h_4^3)$ and $h_1 = h_3(h_3^2 - 4h_2h_4)/(8h_4^2)$ can be found, then

$$\bar{u}(\xi) = -\frac{2h_0k^2}{\varphi(\xi)^2} - \frac{h_1k^2}{\varphi(\xi)} - h_3k^2\varphi(\xi) - 2h_4k^2\varphi(\xi)^2 + \frac{4h_4v^2 + 3h_3^2k^2 - 4h_4 - 16h_2h_4k^2}{24h_4} \tag{4.3}$$

is a solution of the Boussinesq Equation (1.1).

For $h_4 = h_3 = 0$, and $h_1^2 = 4h_0h_2$, by solving the above over-determined system, we obtain the following sufficient conditions for $\bar{u}$ to satisfy (1.1) :

$$c_0 = \frac{v^2 - h_2k^2 - 1}{6},\ c_{-1} = -h_1k^2, c_{-2} = -2h_0k^2,\ c_1 = c_2 = 0 \cdot$$

That is, the Boussinesq Equation (1.1) has solutions of the form

$$\bar{u}(\xi) = -\frac{2h_0k^2}{\varphi(\xi)^2} - \frac{h_1k^2}{\varphi(\xi)} + \frac{v^2 - h_2k^2 - 1}{6} \tag{4.4}$$

where φ is the solution of the ODE (2.1) with coefficients satisfying $h_4 = h_3 = 0$ and $h_1^2 = 4h_0h_2$, and the definition of ξ is as defined in Section 3.

Note that if $h_1 = h_3 = 0$ and $h_2^2 = 4h_0h_4$, then (4.3) reduces to

$$\bar{u}(\xi) = -\frac{2h_0k^2}{\varphi(\xi)^2} + \frac{v^2 - 4h_2k^2 - 1}{6} - 2h_4k^2\varphi(\xi)^2 \cdot \tag{4.5}$$

Noting that ξ is as defined in Section 3, we summarize the above results in the form of the following theorem.

Theorem 4. Suppose that φ is a solution of the ODE (2.1).

(1) If the coefficients $h_i (i = 0, \cdots, 4)$ of the ODE (2.1) satisfy $h_4 \neq 0$, $h_0 = (16h_2^2h_4^2 - 8h_2h_3^2h_4 + h_3^4)/(64h_4^3)$, and $h_1 = h_3(h_3^2 - 4h_2h_4)/(8h_4^2)$, then, for any *k* and *v*,

$$u(\xi) = -\frac{2h_0k^2}{\varphi(\xi)^2} - \frac{h_1k^2}{\varphi(\xi)} - h_3k^2\varphi(\xi) - 2h_4k^2\varphi(\xi)^2 + \frac{4h_4v^2 + 3h_3^2k^2 - 4h_4 - 16h_2h_4k^2}{24h_4}$$

is a solution of the Boussinesq Equation (1.1).

(2) If the coefficients $h_i (i = 0, \cdots, 4)$ of the ODE (2.1) satisfy $h_4 = h_3 = 0$ and $h_1^2 = 4h_0h_2$, then, for any *k* and *v*,

$$u(\xi) = -\frac{2h_0k^2}{\varphi(\xi)^2} - \frac{h_1k^2}{\varphi(\xi)} + \frac{v^2 - h_2k^2 - 1}{6}$$

is a solution of the Boussinesq Equation (1.1).

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Remark 4. Note that the solution form given in Theorem 4 includes (4.5) as a special case.

Now, we give some explicit new exact traveling wave solutions of the Boussinesq Equation (1.1) by using Theorem 4 together with Theorem 1. Note that more solutions of the Boussinesq Equation (1.1) may be obtained by Theorem 4 with Theorem 2 or Theorem 3. For simplify, they are omitted here.

Family 1. Using (4.3) and (2.5), we obtain

$$\bar{u}_1(\xi) = -\frac{32h_2^2h_4k^2}{(e^{\sqrt{h_2}(\beta\xi-c)} - 4\delta\sqrt{h_2h_4})^2} - \frac{8k^2h_2\sqrt{h_2h_4}}{e^{\sqrt{h_2}(\beta\xi-c)} - 4\delta\sqrt{h_2h_4}} + \frac{v^2 - h_2k^2 - 1}{6},$$

where h_2, h_4 and c are arbitrary coefficients, and $h_2 > 0, h_4 \neq 0$, $\beta, \delta \in \{-1,1\}$. Noting that ξ is as defined in Section 3, we get the following exact traveling solutions of the Boussinesq Equation (1.1) after collecting arbitrary coefficients and calculating,

$$u_1(\xi) = -\frac{2a^2k^2c^2}{(be^{a\xi} - c)^2} - \frac{2a^2k^2c\delta}{be^{a\xi} - c} + \frac{v^2 - ak^2 - 1}{6}. \tag{4.6}$$

where *a, b* and *c* are arbitrary constants, and $ac \neq 0, b > 0,$ $= \pm 1$. $k > 0$ is the wave number, *v* is the traveling wave velocity.

Remark 5. Using (4.4) and (2.20), or using (4.4) with (2.21), we get the same solution of the Boussinesq Equation (1.1) as $u_1(t,x)$. Noting $\xi = k(x - vt)$, we have $u_1(t,x) = \bar{u}_1(\xi)$. For $bc < 0$, $u(t,x)$ is continuous on $R^+ \times R$, $\bar{u}_1(\xi)$ is bounded with maximum value $\bar{u}_1(\xi) = a^2k^2/2 + (v^2 - a^2k^2 - 1)/6$ and the maximum value occurs at the point $(\ln(1-2\delta)c - \ln b)/a$. This shows that the Boussinesq Equation (1.1) has smooth solitary solutions or kink solitary solutions. They are shown in Figure 1, and Figure 2.

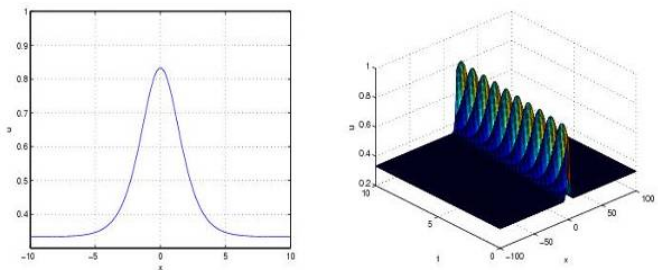

Fig.1 The graph of u_1 with $a = b = k = \delta = 1$, $c = -1$, $v = 2$

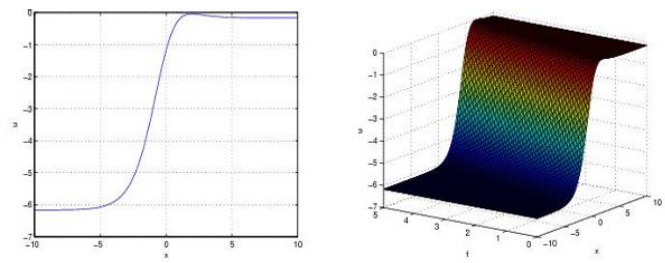

Fig. 2 The graph of u_1 with $a = b = k = 1$, $\delta = c = -1$, $v = 2$

For $bc > 0$, letting $\xi_0 = (\ln c - \ln b)/a \cdot \bar{u}_1(\xi)$ is smooth on either side of ξ_0, and $\lim_{\xi \to \xi_0} \bar{u}_1(\xi) = \lim_{\xi \to \xi_0} \bar{u}_1'(\xi) = \infty$. That is to say, the solution of the Boussinesq Equation (1.1) is unbounded. As shown in Figure 3.

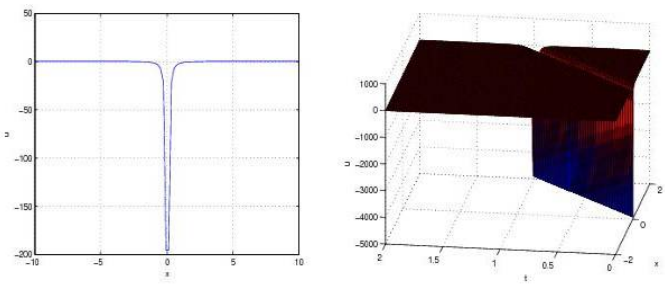

Fig. 3 The graph of u_1 with $a = b = c = k = \delta = 1$, $v = 2$

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Family 2. Using (4.3) with (2.4), we obtain the following traveling solutions of the Boussinesq Equation (1.1):

$$u_2(\xi)=-\frac{8a^2b^2k^2e^{2a\xi}}{(be^{a\xi}+1)^4}-\frac{4a^2bk^2\delta e^{a\xi}}{(be^{a\xi}+1)^2}+\frac{v^2-a^2k^2-1}{6}, \tag{4.7}$$

where a and b are arbitrary non-zero constants, and $\delta=\pm 1$.

Remark 6. We can find easily that u_1 and u_2 have some similar properties. u_2 can be divided into two cases: Case 1 ($b > 0$), bounded solution and there is a maximum or minimum; Case 2 ($b < 0$), unbounded solution. The graphs of u1 is similar to Figure 1 when $b > 0$, to Figure 3 when $b < 0$. We can find easily that u_2 has a peak solitary solution in Case 1.

Family 3. Using (4.3) and (2.7), we get the following solutions of the Boussinesq Equation (1.1):

$$\begin{aligned} u_3(\xi)=&-2a^2k^2(\delta\tanh(a\xi-c)-1)^2-4\delta a^2k^2\tanh(a\xi-c)\\ &+\frac{v^2+20a^2k^2-1}{6}. \end{aligned} \tag{4.8}$$

where a and c are arbitrary constants, and $a\neq 0,\delta=\pm 1$.

Remark 7. $u_3(\xi)$ is bounded, and reaches the maximum value $(v^2+8a^2k^2-1)/6$ when $a\xi=c$ and $\delta=1$, which shows that the solution $u_3(\xi)$ is a smooth solitary solution, and its graph is similar to Figure 1.

Family 4. Using (4.3) and (2.6) leads to the following exact traveling solutions of the Boussinesq Equation (1.1):

$$\begin{aligned} u_4(\xi)=&-2a^2k^2(\delta\coth(a\xi-c)-1)^2+\frac{v^2+20a^2k^2-1}{6}\\ &-4\delta a^2k^2\coth(a\xi-c), \end{aligned} \tag{4.9}$$

where a and c are arbitrary constants, and $\delta=\pm 1$.

Family 5. Substituting (2.11) into (4.3) yields the exact traveling wave solutions of the form:

$$\begin{aligned} u_5(\xi)=&\frac{-8a^2k^2}{\sinh^2(a\xi-c)(2\sinh(a\xi-c)+\delta)^2}-\frac{8a^2k^2}{\sinh(a\xi-c)(2\sinh(a\xi-c)+\delta)}\\ &+\frac{v^2-4a^2k^2-1}{6}, \end{aligned} \tag{4.10}$$

where a and c are arbitrary constants, and $\delta=\pm 1$.

Family 6. Using (4.5) and (2.2), or using (2.18) and (4.4), we get another solution:

$$u_6(\xi)=-\frac{2a^2k^2}{(a\xi-c)^2}+\frac{v^2-1}{6}, \tag{4.11}$$

where a and c are arbitrary constants, and $a > 0$.

Family 7. Combining (2.17) and (4.3) (or (4.5)), we have the following solution of the Boussinesq Equation (1.1):

$$\begin{aligned} u_7(\xi)=&-4a^2k^2\coth^2(a\xi-c)-4a^2k^2\tanh^2(a\xi-c)\\ &+\frac{v^2+16a^2k^2-1}{6}, \end{aligned} \tag{4.12}$$

where a and c are arbitrary constants.

Remark 8. u_4, u_5, u_6, and u_7 are unbounded, since their denominator may be zero. Their graphs are similar to Figure 2.

Family 8. Combining (4.3) and (2.8), we have the following solutions of the Boussinesq Equation (1.1):

$$\begin{aligned} u_8(\xi)=&-\frac{8a^2k^2}{(\sinh(2a\xi-c)-2\delta\cosh^2(a\xi-c))^2}\\ &+\frac{8a^2k^2\delta}{\sinh(2a\xi-c)-2\delta\cosh^2(a\xi-c)}\\ &+\frac{v^2-4a^2k^2-1}{6}, \end{aligned} \tag{4.13}$$

where a and c are arbitrary constants, and $\delta=\pm 1$.

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Remark 9. Using (4.3) with (2.12), we have the same result as u_8. From the expression of the solution ¯u8, it is easy to find that $\bar{u}_8$ is continuous and bounded, since the denominator in the expression can not be zero at any point, and

$$\lim_{\xi\to\infty}\bar{u}_8(\xi)=\begin{cases}\dfrac{v^2-4a^2k^2-1}{6}, c\neq 0,\\ \dfrac{v^2-100a^2k^2-1}{6}, c=0.\end{cases}$$

Clearly, the graphs of u_8 are similar to Figure 1. It is a peak solution.

Family 9. Combining (2.16) and (4.3) (or (4.5)), we have the following solution of the Boussinesq Equation (1.1):

$$u_9(\xi)=-4a^2k^2\tan^{-2}(a\xi-c)-4a^2k^2\tan^2(a\xi-c))+\frac{v^2-16a^2k^2-1}{6}, \tag{4.14}$$

where a and c are arbitrary constants, and $a > 0$.

Remark 10. This is a kind of periodic solutions. It is unbounded since its denominator can be zero. As shown in Figure 4.

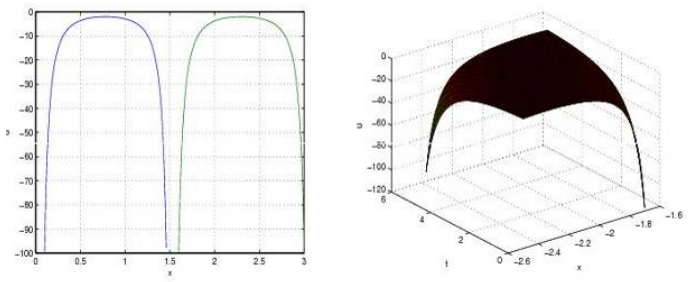

Fig. 4 The graph of u_9 with $c = 0$, $k = 1$, $a = 0.7$, $v = 0.1$

Family 10. Using (2.9) with (4.3) and (2.10) with (4.3) respectively, we obtain the following solution of the Boussinesq Equation (1.1):

$$u_{10}(\xi)=\frac{8a^2k^2}{(2\cos^2(a\xi-c)+i\sin(2a\xi-2c))^2}-\frac{8a^2k^2}{2\cos^2(a\xi-c)+i\sin(2a\xi-2c)}+\frac{v^2+4a^2k^2-1}{6}, \tag{4.15}$$

$$u_{11}(\xi)=\frac{8a^2k^2}{(2\sin^2(a\xi-c)+i\sin(2a\xi-2c))^2}-\frac{8a^2k^2}{2\sin^2(a\xi-c)+i\sin(2a\xi-2c)}+\frac{v^2+4a^2k^2-1}{6}, \tag{4.16}$$

where a and c are arbitrary constants, and $\delta=\pm 1$, $i^2=-1$.

V. CONCLUSION

We have successfully found many new classes of exact traveling wave solutions of the Boussinesq equation by using a generalized mapping method and the new solutions of the auxiliary equation. Most of the new solutions cannot be obtained by other direct methods. More importantly, this method can be applied to many other nonlinear differential equations in physics and nonlinear mechanics.

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