Fluctuation Scaling and 1/*f* Noise

Shared Origins from the Tweedie Family of Statistical Distributions

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***Abstract*-A power law relationship between the variance and the mean, when derived from sequential data using expanding enumerative bins, implies 1/*f* noise. This relationship, called fluctuation scaling by physicists and Taylor's law by ecologists, is found within diverse physical, econometric and biological systems. Its origin remains controversial. Both fluctuation scaling and 1/*f* noise are proposed to manifest consequent to a central limit-like effect specified by the Tweedie convergence theorem that has as its foci of convergence a family of statistical distributions, the Tweedie exponential dispersion models. An example of fluctuation scaling and 1/*f* noise is provided here based on deviations in position of the prime numbers; the Tweedie compound Poisson distribution is shown to correspond to these deviations. Whereas many different physical and biological mechanisms have been proposed to explain fluctuation scaling, Taylor's law and 1/*f* noise, such mechanisms are inapplicable to a number theoretic example like this. The Tweedie convergence theorem provides a generally applicable explanation for the origin of these scaling relationships, and can provide insight into processes like self-organized criticality and multifractality.**

Keywords- Taylor's Power Law; Exponential Dispersion Models; Multifractal; Self-organized Criticality; Prime numbers

I. INTRODUCTION

Complex dynamical systems often manifest scaling behavior like 1/*f* noise, characterized by power spectra of the form $S(f) \propto 1/f^{\gamma}$ with $0 < \gamma < 2$ and *f* representing frequency. Another related scaling behavior, fluctuation scaling[1], is also characterized by a power law relationship, now between the variance of a signal $\mathrm{Var}(n)$ and its mean $\mathrm{E}(n)$ such that,

$$\mathrm{Var}(n) \propto \mathrm{E}(n)^{b}, \tag{1}$$

where the exponent *b* is a real-valued positive constant. This latter relationship manifests within physical and econometric systems [1, 2] and, as well, has been observed within many biological systems [3] where it has been called Taylor's power law [4].

The origins of these scaling relationships remain a matter of controversy. A widely held paradigm for 1/*f* noise involves a dynamical mechanism called self-organized criticality[5]; in the physics literature fluctuation scaling has been attributed to a convergence property called impact inhomogeneity[1] as well as to the influence of an external physical field within non-equilibrium statistical mechanical systems[2]. Biologists have explained Taylor's law variously in terms of the balance between the migratory and congregatory behavior of animals [6], random demographic effects within populations [7], environmental stochasticity [8], and interspecies interactions [9].

Recently another hypothesis has been proposed to explain Taylor's law [3] based on mathematical convergence effects associated with the Tweedie exponential dispersion models, a family of statistical models used to model error distributions from the general linear model [10, 11]. This Tweedie hypothesis has been extended to explain the origin of 1/*f* noise [12] and has been found applicable to such scaling phenomena from number theory [13] and random matrix theory [12]. Here, a brief introduction to the Tweedie exponential dispersion models will be provided along with an additional number theoretic example of fluctuation scaling and 1/*f* noise, found within the distribution of prime numbers. Numerical examples like this are of interest: they cannot be explained by physical or biological mechanisms, yet the applicable mathematical theory can yield mechanistic insight into related physical and biological processes.

II. THE DOUBLE POWER LAW

The eponym Taylor's law has been applied to different manifestations of the variance to mean power law (Eq. 1). In Taylor's initial description, the clustering of animals and plants within their habitats was assessed by dividing their habitats into a set of equal-sized and non-overlapping rectangular quadrats and, though multiple enumerative samples drawn from each quadrat, the mean and variance of the number of individuals of a species were calculated and plotted on log-log graphs [14]. A straight line relationship on a log-log plot of variance versus the mean was used to infer a power law; this finding has been confirmed for many hundreds of animal species [4, 6, 14]. Values of the exponent *b* greater than 1 have been taken to indicate a non-random clustering of organisms, compared to $b = 1$ which indicates a Poisson, or random, distribution. As we shall see below the distinction between what is random and non-random is not so clear.

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The variance to mean power law has also been observed from time series data [9] and other sequential data [15, 16], commonly though a different enumerative method. In the method of expanding bins, a sequence of numerical measurements is apportioned into equal-sized and non-overlapping enumerative bins, the data values within each bin are summed, and the mean and variance of the new sequence of summed values are determined, and re-determined over a range of different bin sizes. A log-log plot of these variances versus their corresponding means may demonstrate a straight line relationship indicative of the power law Eq. 1 with appropriate processes.

By this second method, the sequential distribution of genes [16] and single nucleotide polymorphisms [15] within the human genome, as well as the spatial distribution of insects [17] and temporal manifestation of measles epidemics [18] have been represented as having demonstrated Taylor's law. Although both Taylors's original method, and the method of expanding bins, both can yield variance to mean power laws, the associated scaling behaviours are mathematically distinct.

A double power law has been proposed to account for the scaling properties of these two approaches [19]. We define the mean of the number of individuals per unit area $\overline{\mu}$ of habitat and the mean number of individuals per sampling bins of size *t*, $\mathrm{E}(n) = \overline{\mu}\, t$. The double power law is then

$$\mathrm{Var}(n) = \overline{a}\,\overline{\mu}^{b} t^{2-\beta} \tag{2}$$

with $\overline{a}$ being a proportionality constant and β a dimensional parameter related to fractal dimension [19]. This equation distinguishes the scaling behaviour of the mean number of individuals per unit area from the mean number per sampling bin. Despite this difference, one may still plot a variance to mean power function with either mean, $\overline{\mu}$ or E(*n*) provided appropriate data.

III. SELF-SIMILAR PROCESSES

We will focus here on the scaling properties found with the method of expanding bins, which have been well studied in the context of self-similar, or fractal, processes [20]. The mathematics of these processes was developed to describe the self-similarity or "burstiness" of Ethernet traffic [20].

Given the discrete time series sequence $Y = (Y_i : i = 0, 1, 2, ..., N)$ with mean $\hat{\mu} = \mathrm{E}[Y_i]$, any deviations about the mean $y_i = Y_i - \hat{\mu}$ can be assessed using the autocorrelation function

$$r(k) = \mathrm{E}[y_i y_{i+k}] / \mathrm{E}[y_i^2] \tag{3}$$

defined for the lag *k*. The variance of the sequence is

$$\mathrm{var}[Y] = \mathrm{var}[y] = \hat{\sigma}^2 = \mathrm{E}[y_i^2]\,. \tag{4}$$

Self-similar processes are defined based on their long-range property[21],

$$r(k) \sim k^{-\beta} L(k), \quad k \to \infty \tag{5}$$

with $0 < \beta < 1$ being a real-valued constant and $L(k)$ being a slowly varying function as $k \to \infty$. We can construct a set of equal sized counting bins of length *m* that can be used to further define a set of sequences $Y^{(m)}$ such that

$$Y_i^{(m)} = \frac{1}{m}(Y_{im-m+1} + ... + Y_{im})\,, \quad i > 1, \tag{6}$$

where *m* is an integer chosen so that *N/m* is also an integer. The mean $\hat{\mu} = E[Y] = E[Y^{(m)}]$ and variance $\hat{\sigma}^2$ of *Y* are held to be constants, and so the variance to mean power law

$$\mathrm{var}[Y^{(m)}] = \hat{\sigma}^2 m^{-\beta} \tag{7}$$

can be specified if and only if [22]

$$r(k) = \frac{1}{2}[(k+1)^{2-\beta} - 2k^{2-\beta} + (k-1)^{2-\beta}]\,. \tag{8}$$

This autocorrelation function also obeys the limit [22]

$$\lim_{k\to\infty} \frac{r(k)}{k^{-\beta}} = \frac{1}{2}(2-\beta)(1-\beta) = H(2H-1)\,, \tag{9}$$

where *H* is the Hurst parameter.

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There corresponds to these reproductive sequences $Y^{(m)}$ a parallel set of additive sequences

$$Z_i^{(m)} = (Y_{im-m+1} + \ldots + Y_{im}) \tag{10}$$

with means and variances $E[Z^{(m)}] = mE[Y^{(m)}]$ and $\mathrm{var}[Z^{(m)}] = m^2\,\mathrm{var}[Y^{(m)}]$, respectively. Provided that $\hat{\mu}$ and $\hat{\sigma}^2$ are constants we can construct a variance to mean power law from this method of expanding bins,

$$\mathrm{var}[Z_i^{(m)}] = m^2\,\mathrm{var}[Y^{(m)}] = (\hat{\sigma}^2 / \hat{\mu}^{2-\beta})E[Z_i^{(m)}]^{2-\beta} \tag{11}$$

where the exponent $b = 2 - \beta$. The biconditional relationship between Eq. (6) and Eq. (7) implies that any sequence that yields this variance to mean power law will also manifests autocorrelation functions that have the limiting form $r(k) \propto k^{-\beta}$.

The power spectral density $S(f)$ can be obtained from the autocorrelation via Fourier transform,

$$S(f) = \int_{-\infty}^{\infty} r(k) e^{-2\pi i f k} dk\,. \tag{12}$$

We have, from the Wiener-Khintchine theorem [23], the relationship $S(f) \propto f^{\beta-1}$. Power spectra that take this form with $0 < 1 - \beta < 2$ demonstrate 1/*f* noise. Thus sequential data that demonstrate 1/*f* noise should also be expected to manifest a variance to mean power law, and vice versa.

1/*f* noise was described initially from time sequence data; in more recent years the term has been applied to such scaling associated with any form of discrete sequential data. In these latter cases a frequency analogue is inferred from the sequential data as is done with time series data.

IV. THE TWEEDIE MODELS

Exponential dispersion models are a class of probability distributions developed to describe error distributions for generalized linear models [11]. A subclass of these models, known as the Tweedie models [10], are defined by closure under additive and reproductive convolution as well as under scale transformation [11]. The reproductive exponential dispersion models $\mathrm{ED}(\mu, \sigma^2)$, specified by position parameter μ and dispersion parameter σ, subject to a change in scale by the factor *c* will necessarily obey the transformation rule,

$$c \cdot \mathrm{ED}(\mu, \sigma^2) = \mathrm{ED}(c\mu, c^{2-b}\sigma^2)\,. \tag{13}$$

This rule implies that the variance function $V(\mu)$ relates to the position parameter such that $V(\mu) = \mu^b$.

In the case of additive exponential dispersion models defined for the random variable *Z*, the variance $\mathrm{Var}(Z) = \lambda V(\mu)$ and the mean $\mathrm{E}(Z) = \lambda\mu$ are related by the equation $\mathrm{Var}(Z) = a \cdot \mathrm{E}(Z)^b$ where $\lambda = 1/\sigma^2$.

The Tweedie models are conventionally sub-classified in accordance with the values of the exponent *b*. For the additive Tweedie models we have the cumulant generating functions (CGFs):

$$K_b^*(s;\theta,\lambda) = \begin{cases} \lambda e^{\theta}\left(e^s - 1\right) & b = 1 \\ \lambda \kappa_b(\theta)\left[(1 + s/\theta)^{\alpha} - 1\right] & b \neq 1,2\,, \\ -\lambda \log\left(1 + s/\theta\right) & b = 2 \end{cases} \tag{14}$$

where θ and λ are the canonical and index parameters, *s* is the generating function variable, the exponent $\alpha = (b-2)/(b-1)$, and the cumulant function $\kappa_b(\theta)$ is

$$\kappa_b(\theta) = \begin{cases} e^{\theta} & b = 1 \\ \dfrac{\alpha - 1}{\alpha}\left(\dfrac{\theta}{\alpha - 1}\right)^{\alpha} & b \neq 1, 2\,. \\ -\log(-\theta) & b = 2 \end{cases} \tag{15}$$

We have the extreme stable distributions for $b < 0$; the Gaussian distribution, $b = 0$; the Poisson distribution, $b = 1$; the compound Poisson distribution, $1 < b < 2$; the gamma distribution, $b = 2$; the positive stable distributions, $2 < b < 3$; the inverse Gaussian distribution, $b = 3$; the positive stable distributions, $b > 3$; the extreme stable distributions, $b = \infty$; and for

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$0 < b < 1$ the Tweedie models do not exist.

Whereas many of the Tweedie models include well-known distributions like the Gaussian, Poisson and gamma distributions, there are other distributions that are less common. The Tweedie compound Poisson distribution is one of these; data conforming to it represent a random sequence of discrete 'jumps' in value. A closed form expression for the compound Poisson probability density currently does not exist. It can, however, be expressed in the form [11]:

$$p^*(z;\theta,\lambda,\alpha) = c^*(z;\lambda)\exp[\theta \cdot z - \lambda\kappa(\theta)] \tag{16}$$

where

$$c^*(z;\lambda) = \begin{cases} \frac{1}{z}\sum_{n=1}^{\infty} \lambda^n \kappa^n (-1/z) \big/ \Gamma(-\alpha \cdot n) n! & \text{for z} > 0 \\ 1 & \text{for } z = 0 \end{cases} . \tag{17}$$

Means and variances can be derived from the CGFs by differentiation with respect to s and setting $s = 0$. This exercise can be used to confirm that the power law $\mathrm{Var}(Z) = a \cdot \mathrm{E}(Z)^b$ follows as an inherent property of the Tweedie distributions.

The relationship between 1/*f* noise and the variance to mean power law would suggest the corresponding range of the power law exponent b should fall within the interval (1, 3). This would implicate the compound Poisson, gamma and positive stable distributions as potential candidates for stochastic processes manifesting 1/*f* noise.

The compelling reason to propose the Tweedie models as explanations for fluctuation scaling/Taylor's law and 1/*f* noise rests with their role as foci of convergence in a central limit-like effect. The Tweedie convergence theorem states that for exponential dispersion models $\mathrm{ED}(\mu,\sigma^2)$ with unit variance functions and of the form $V(\mu) \sim \mu^b$, as $\mu \to 0$ or $\mu \to \infty$ then $c^{-1}\mathrm{ED}(c\mu,\sigma^2 c^{2-b})$ will converge to the form of a Tweedie model [24] as the factor $c \downarrow 0$ or $c \to \infty$. Indeed, the variance functions of many probability distributions will approximate $V(\mu) \propto \mu^p$ as $\mu \to 0$ or $\mu \to \infty$, and so these models have their role as foci of convergence for a wide variety of stochastic processes.

The Tweedie convergence theorem can be viewed as part of a spectrum of limit theorems, which would include the Poisson convergence theorem for $b = 1$ and the central limit theorem for $b = 0$. Other convergence theorems may be proposed to account for fluctuation scaling or 1/*f* noise. However limit theorems for independent and identically distributed random variables, like the Tweedie convergence theorem, should be considered as being primary for the generation of any given distributional form.

V. CHEBYSHEV DEVIATIONS

Self-similar processes that demonstrate both a variance to mean power law and 1/*f* noise have been reported with the eigenvalue deviations of the Gaussian Unitary Ensemble (GUE) and Gaussian Orthogonal Ensemble (GOE) of random matrix theory[12]. We will examine here another type of deviation associated with the placement of prime numbers amongst the natural numbers. We will begin with a brief introduction to prime number theory.

Gauss has been credited for estimating that the local density of prime numbers near the *n*th prime p_n should be approximately $1/\ln(p_n)$. Integration of this density gives an average number of prime numbers less than a given value *x*, which is known as the logarithmic integral

$$\mathrm{Li}(x) = \int_0^x \frac{dt}{\ln(t)} . \tag{18}$$

The exact number of primes less than the real value *x* is given by the prime counting function $\pi(x)$, and so a measure of prime number deviations may be obtained from

the difference $\mathrm{Li}(x) - \pi(x)$. Riemann has provided an explicit estimate for this difference [25] ,

$$\mathrm{Li}(x) - \pi(x) \sim -\sum_{n=2}^{\infty} \frac{\mu(n)}{n} \mathrm{Li}(x^{1/x}) , \tag{19}$$

where $\mu(n)$ is the Möbius function on the integers *n*,

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$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n \text{ has 1 or more repeated prime factors} \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes} \end{cases} . \tag{20}$$

The fact that this difference tends asymptotically towards zero for large values of x follows from the prime number theorem, proven independently by Hadamard [26] and de la Vallée Poussin [27]. The error terms in this theorem can be related to the Riemann hypothesis that the nontrivial zeros of the Riemann zeta function on the complex variable s

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{21}$$

have as their real part the value $\mathrm{Re}(s) = 1/2$. If the Riemann hypothesis is true, Schoenfeld has shown that [28]

$$\left|\pi(x) - \mathrm{Li}(x)\right| < (\sqrt{x} \cdot \mathrm{Log}(x))/8\pi , \tag{22}$$

for all $x \geq 2657$.

The prime counting function $\pi(x)$ can alternatively be replaced by the Chebyshev function

$$\psi(x) = \sum_{n \leq x} \Lambda(n) , \tag{23}$$

which has the advantage of more linear dependence on x. Here, $\Lambda(x)$ is the von Mangoldt function which takes the value of $\log(p)$ if x is a positive power of a prime number p, but otherwise assumes the value of 0. The expression $\psi(x) \sim x$ is then equivalent to the prime number theorem and we have, if the Riemann hypothesis is true, the relationship [28]

$$\left|\psi(x) - x\right| < \sqrt{x} \log^2(x)/(8\pi) \quad \text{for } x \geq 73.2 . \tag{24}$$

We will now focus on the Chebyshev deviations,

$$\Delta(x) = \left|\psi(x) - x\right| . \tag{25}$$

Fig. 1 provides these deviations for the first thousand (1a) and million (1b) integer positions. The patterns from both graphs appeared self-similar. There was a similar increase in size of deviations over the first million values as seen with the first thousand. The pattern of these deviations was also characterized by irregular cusps (multifractal singularities) as well by segments where the deviations within a certain range of magnitude were irregularly punctuated by other segments with an abruptly different magnitude. To a degree these data could be considered to have lacked stationarity. The analytical methods being applied here, however, were sufficiently robust and the scaling properties sufficiently strong that certain features remained evident from these data.

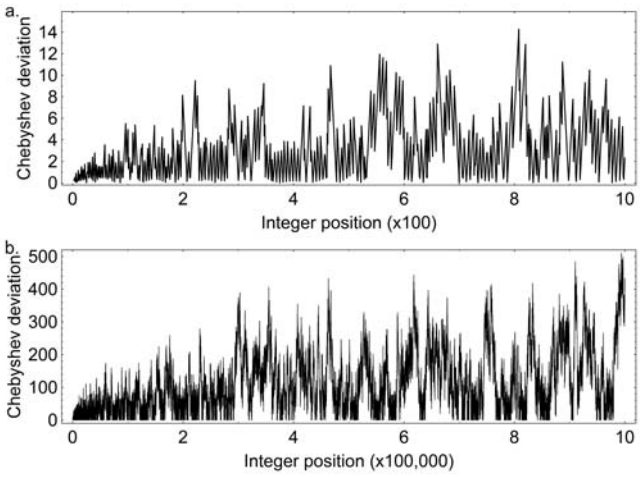

Fig. 1 Chebyshev deviations $\Delta(x)$ for the first thousand (a), and million (b), integer positions

The Chebyshev deviations were evaluated empirically using the method of expanding bins, over a range of bin sizes from 1 to 25,000 integer positions. A log-log plot of the means and variances was then constructed (Fig. 2). The plot revealed an approximately linear fit with the power law exponent b = 1.93. The fit was quite close, holding over several orders of magnitude of the mean, yet some discrepancies from this linear fit were evident at the extremes.

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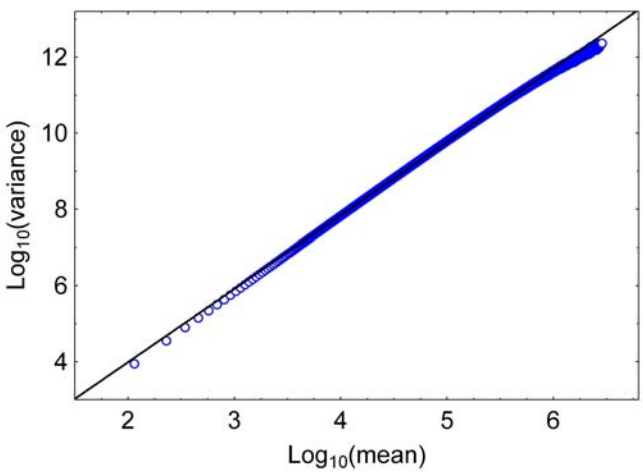

Fig. 2 Variance to mean power law function for Chebyshev deviations

To further analyse the properties of b the data set was divided into sequential non-overlapping segments of 10^3 integers and a variance to mean power function was fitted from the data contained within each of these smaller segments. Fig. 3 provides the frequency histogram for the measured values of b. A range of values was found from 1.87 up to just over 2.0.

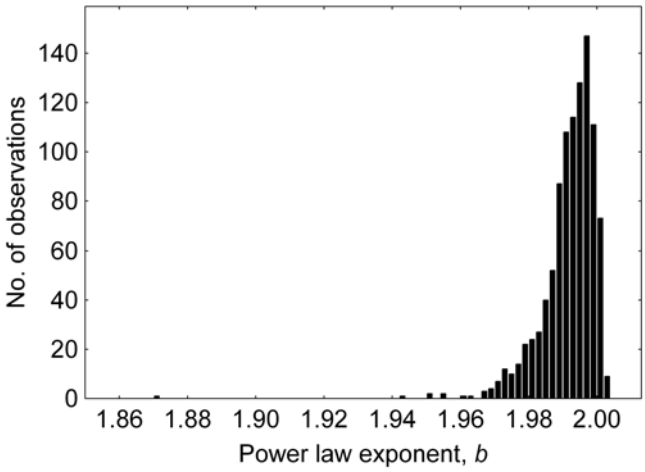

Fig. 3 Frequency histogram for the power law exponent, b

We will return to the implications of this distribution of values for b later. Before doing this a second evaluation of the applicability of the Tweedie models was conducted. The empirical cumulative distribution function (CDF) derived from the deviations $\Delta(x)$ was fitted to the theoretical Tweedie compound Poisson CDF. Fig. 4 provides the probability-probability plot.

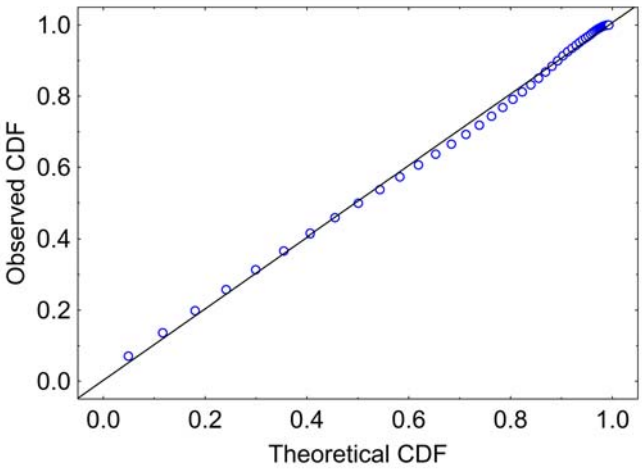

Fig. 4 Probability-probability plot for the Chebyshev deviations based on the Tweedie compound Poisson CDF

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This plot revealed an approximately linear relationship, which indicated an acceptable model fit. The value for the power law exponent obtained from the theoretical CDF was b = 1.81, reasonably close to those obtained from the variance to mean power law in Fig. 2.

Next the Chebyshev deviations were examined to see whether they might reveal $1/f$ nose. In this evaluation the first 100,000 values of $\Delta(x)$ were subject to a discrete Fourier transform using a Hamming window of 555 data positions to smooth the resultant spectrum. Fig. 5 provides a log-log plot of power versus frequency.

The power spectrum revealed a region of noise with an

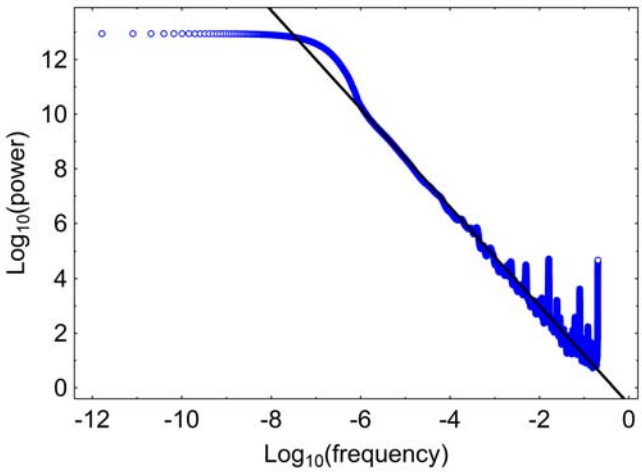

Fig. 5 Power spectrum for the Chebyshev deviations $\Delta(x)$

Approximate $1/f^{1.8}$ dependence that ran over several orders of magnitude of frequency. This $1/f$ noise pattern corresponded to a predicted value of b = 2.8.

Although a variance to mean power law could be demonstrated from the Chebyshev deviations over several orders of magnitude, the empirical CDF and the Tweedie compound Poisson CDF appeared consistent, and $1/f$ noise could be detected, some discrepancy was evident from the measured values for b from the different assessments. Indeed, the analysis connected to Fig. 3 had demonstrated that the power law exponent b possessed a distribution of values. We recall also that Fig. 1 had demonstrated both a lack of stationarity as well as multifractal-like singularities and abrupt transitions in the magnitude of the fluctuations. These properties could have contributed to a bias in estimate for b, particularly with that made in the frequency domain [29].

Multifractal analysis has evolved to deal with some of the inadequacies of more conventional methodologies in the analysis of complex signals[30]. The wavelet transform method of multifractal analysis will be applied here to the deviations $\Delta(x)$.

To briefly introduce the technique one may construct a wavelet transform $T_\Psi[\tilde{f}](\tilde{b},\tilde{a})$ of a function $\tilde{f}$ by means of decomposition with analysing wavelets Ψ, for translations and dilations specified by the scale and shape parameters $\tilde{a} \in \mathbf{R}^+$ and $\tilde{b} \in \mathbf{R}$:

$$T_\Psi[\tilde{f}](b,a) = \frac{1}{\tilde{a}}\int_{-\infty}^{+\infty}\Psi\left(\frac{x-\tilde{b}}{\tilde{a}}\right)\tilde{f}(x)dx\,. \tag{26}$$

Typically successive differentiations of the Gaussian function are used to generate analysing wavelets,

$$\Psi^{(N)}(x) = d^N(\mathrm{e}^{-\mathrm{x}^2/2})/dx^N\,. \tag{27}$$

Multifractal singularities within the function $\tilde{f}$ at the point x_0 are assessed by means of the local Hölder exponent $h(x_0)$, the largest valued exponent for which a polynomial $P_n(x)$ of order n exists that satisfies

$$\left|f(x) - P_n(x-x_0)\right| = O(\left|x-x_0\right|^h) \tag{28}$$

for x in a neighborhood of x_0. The $D(h)$, or singularity, spectrum provides the Hausdorff dimension where the Hölder exponent takes the value h,

$$D(h) = \dim_H[x|h(x)=h] \tag{29}$$

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A partition function

$$Z(a,q) = \sum_{l \in \mathcal{L}(a)} \left\{ \sup_{(x,a') \in l} \left| T_\psi[\tilde{f}](x,a') \right| \right\}^q \tag{30}$$

can then be constructed such that $q \in \mathbf{R}$ specifies the order of the generalized fractal dimension, and $\mathcal{L}(a)$ represents the set of wavelet maxima lines l_i that reach or cross the a-scale. If the partition function $Z(a,q) \sim a^{\tau(q)}$ demonstrates power law scaling, one may then construct a multifractal spectrum from the exponents $\tau(q)$. These exponents $\tau(q)$ are related to the singularity spectrum $D(h)$ by the Legendre transformation,

$$D(h) = \min_q [qh - \tau(q)] . \tag{31}$$

A wavelet analysis was conducted on the initial 10,000 values of the Chebyshev deviations. Fig. 6 gives the resultant multifractal spectrum.

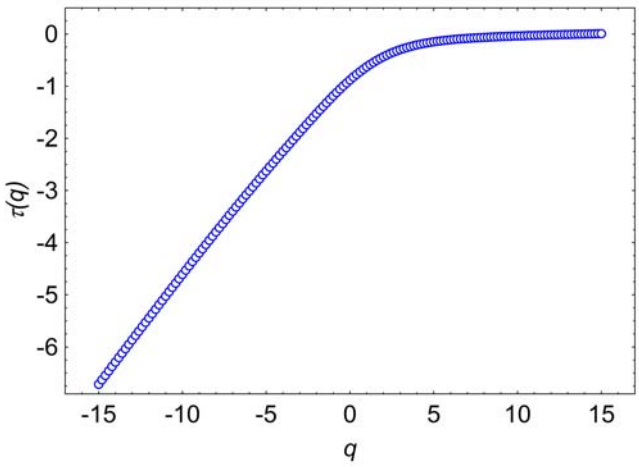

Fig. 6 Multifractal spectrum for the Chebyshev deviations Δ(x)

This spectrum exhibited an inflexion in its slope, indicative of multifractality. The corresponding singularity spectrum is given in Fig. 7.

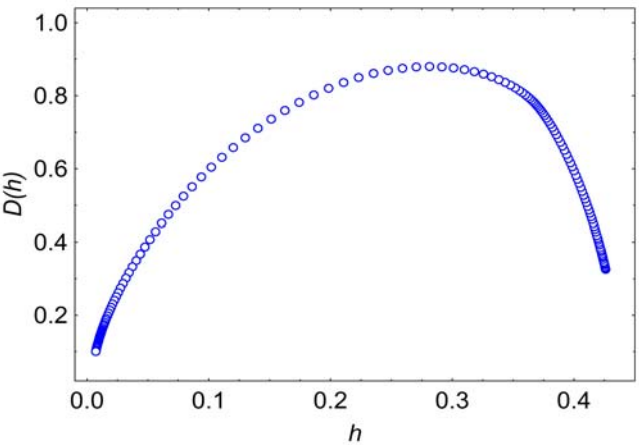

Fig. 7 Singularity spectrum for the Chebyshev deviations Δ(x)

This second spectrum had an inverted convex form that also indicated the presence of multifractality within the Chebyshev deviations Δ(x).

VI. CONCLUSIONS

The analysis presented here provided reasonable evidence that fluctuation scaling and 1/f noise could be found with the Chebyshev deviations Δ(x). The variability evident to the assessed values of b could be attributed to the lack of stationarity in the sequence Δ(x), to limitations of the assessment methods, and to an inherent multifractality. Despite these discrepancies the

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Tweedie compound Poisson model corresponded fairly well to these data, much as might be expected from the Tweedie convergence theorem.

The power law exponent b relates to the Hurst parameter H and the fractal dimension D through the equation,

$$D = 2 - H = 2 - b/2 \,. \tag{32}$$

As demonstrated within Fig. 3 the sequence of Chebyshev deviations revealed an apparently random distribution of values over its assessed length that would correspond to local variations in D. When different regions of an object are found to exhibit different fractal properties the term multifractal can be applied [31].

There are various criteria that have been used to identify multifractals; often the determination is based on the presence of cusps in the data sequence (Fig. 1) in conjunction with a typical inflection seen in the multifractal spectrum (Fig. 6) and convex form seen in the singularity spectrum on wavelet analysis (Fig. 7).

If we accept all of these demonstrations of multifractality within the Chebyshev deviations, and the evidence that the Tweedie distributions provide a model for these deviations (Fig. 4) there is a further implication. The exponent $\alpha = (b-2)/(b-1)$ from the Tweedie CGFs (Eqs. 14, 15) would similarly be affected by the variation associated with b. Since the Tweedie convergence theorem specifies that the Tweedie exponential dispersion models act as foci for the mathematical convergence of a broad range of complicated systems, we would have a plausible mechanism for the genesis of multifractality.

Multifractality is attributed to long range temporal correlations or fat-tailed probability distributions. In the case of the Chebyshev deviations long range temporal correlations were demonstrated by virtue of the demonstration of $1/f$ noise. However, the Tweedie convergence theorem provides additional mathematical insight into multifractality that has not been well recognized.

The Chebyshev deviations served the purpose to provide a further example of fluctuation scaling and $1/f$ noise. Because these deviations were purely mathematical in origin, any physical mechanism used to explain fluctuation scaling and any of the biological explanations for Taylor's power law should be considered clearly inapplicable. Similarly, the dynamical mechanism proposed in self-organized criticality should also be considered inapplicable. Other purely mathematical examples of fluctuation scaling have been reported, most notably with the GUE and GOE [12]. In contrast, the Tweedie convergence theorem with its foci of convergence the Tweedie exponential dispersion models provided a generally applicable explanation for the origin of fluctuation scaling and $1/f$ noise.

The Tweedie convergence theorem states that any exponential dispersion model, which has an asymptotical power law variance function, will have as its domain of attraction a Tweedie model. Since any distribution function that has a finite CGF constitutes an exponential dispersion model, the range of distributions that can be approximated by the Tweedie models is very large, and thus this convergence theorem covers a wide range of processes.

Processes that consist of multiple small independent perturbations would likely tend to converge to the form of a Tweedie model. This would include many of the numerical simulations and algebraic approximations for Taylor's law as well as simulations for self-organized criticality.

The biconditional relationship between Eqs. 7 and 8 in self-similar processes (Section III) provides a theoretical connection between fluctuation scaling and $1/f$ noise. Provided this relationship the Tweedie convergence theorem can then be viewed as the basis for the genesis of $1/f$ noise.

Bak, Tang and Wiesenfeld have proposed that $1/f$ noise can be attributed to self-organized criticality, where critical states can spontaneously arise within extended dynamical systems that possess many degrees of freedom [32]. These critical states characteristically have no intrinsic length or time scales and so, long range temporal correlations and $1/f$ noise could be viewed related to an associated fractal structure of the critical state[33]. We have seen here how long range correlations and $1/f$ noise associated with the Chebyshev deviations can be attributed to a mathematical convergence theorem related to the Central Limit Theorem. Further evidence for such convergence and scaling behavior has been demonstrated with the GUE and GOE [12] as well as with the Mertens function [13].

All of these examples are purely numerical and cannot be attributed to physical processes like self-organized criticality. Consider, for example the derivation of the Maxwell-Boltzmann distribution: using physical principles the associated entropy is maximized subject to the requirement that the system's kinetic energy be proportional to the absolute temperature to yield this distribution. On the other hand one may apply the Central Limit Theorem to the momentum exchange distributions of the particles in the gas to yield the same distribution. The Central Limit Theorem, though, has applicability and generality that ranges further than the *ad hoc* application of physical principles [13].

In this context the Tweedie Convergence Theorem provides an explanation for the origin of the scaling behavior seen with all of the examples provided and referred to here. Self-organized criticality can still be regarded to explain sand piles and avalanches; however, the potential exists here that the Tweedie Convergence Theorem and its foci of convergence, the Tweedie exponential dispersion models might be applicable here, too.

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