Expectation Value in Bell's Theorem

Zheng-Chuan Wang

Department of Physics, Graduate School of the Chinese Academy of Sciences

P. O. Box 4588, Beijing 100049, China

wangzc@gucas.ac.cn

Abstract- **Aspect et al.'s experiment[5] verified the violation of Bell inequality and the validity of quantum mechanics rather than local hidden-variables theory. In this manuscript, we will show that there exists difference between definitions for the expectation value using the probability distribution in Bell's inequality and the expectation value in quantum mechanics. It is this difference that leads to the controversy between the expectation values in quantum mechanics and Bell's inequality. The expectation values in quantum mechanics should obey their own inequality.**

Keywords- Bell's inequality; Expectation value; Local hidden-variables theory

I. INTRODUCTION

Local hidden-variables theory[1] was ever a striking point in the development of quantum mechanics, yet it's validity was questioned by Bell. In 1964, Bell proposed a famous inequality[2] (or theorem) and asserted that local hidden-variables theory conflicts with quantum mechanics and can't reproduce all the prediction of the latter. This theorem was latterly improved to a new version after Clauser et al.'s experiment[3] of testing local hidden-variables theory, which is suit for the entire family of deterministic and nondeterministic local hidden-variables theory[4]. From then on, Bell's inequality had been discussed widely whatever in theories or experiments. In 1982, Aspect et al. [5] proposed an experiment to test Bell's inequality by use of time-varying analyzers, their experimental results coincide with the prediction of quantum mechanics and indicate Bell's inequality is violated. That is a serious challenge to local hidden-variables theory. However, the debates never stopped[6]. Jaynes ever criticized that the probabilistic reasoning in Bell's theorem does not follow the rules of probability theory[7]. Fine discussed the joint distributions and commtativity in Bell theorem[8]. There are other viewpoints on Bell's theorem[9,10], too. In this manuscript, we will demonstrate that the difference of definitions for the expectation value in Bell's inequality and the expectation value in quantum mechanics results in the conflicts, the expectation value in quantum mechanics doesn't satisfy the Bell's inequality.

II. THE DIFFERENCES BETWEEN EXPECTATION VALUES

Consider two Particles A and B which have spin 1/2 and are in quantum mechanical state $|\psi>_{AB}$. 'a' and 'b' are vectors in ordinary three-space, $A(a,\lambda)$ is the outcome of a measurement on $\sigma_A \cdot a$ and $B(b,\lambda)$ is on $\sigma_B \cdot b$, then the expectation value in Bell's inequality is

$$P(a,b) = \int d\lambda \rho(\lambda) A(a,\lambda) B(b,\lambda) \tag{1}$$

where $\rho(\lambda)$ is the probability distribution in local hidden-variables theory. We can't identify it with the expectation value

$${}_{AB}\!<\psi|(\sigma_A \cdot a)(\sigma_B \cdot b)|\psi>_{AB}$$

in quantum mechanics as usual. In fact, in local hidden-variables theory the latter can be written as:

$$\begin{aligned} &{}_{AB}\!<\psi|(\sigma_A \cdot a)(\sigma_B \cdot b)|\psi>_{AB} = \\ &d\lambda \rho(\lambda)\, {}_{AB}\!<\psi|(\sigma_A \cdot a)(\sigma_B \cdot b)|\psi>_{AB}(\lambda) \end{aligned}, \tag{2}$$

while the expectation value $P(a,b)$ in Bell's theorem is:

$$\begin{aligned} P(a,b) = &\int d\lambda \rho(\lambda)\, {}_{AB}\!<\psi|(\sigma_A \cdot a)|\psi>_{AB}(\lambda)\, {}_{AB}\times \\ &<\psi|(\sigma_B \cdot b)|\psi>_{AB}(\lambda) \end{aligned}, \tag{3}$$

where we have replaced the outcomes $A(a,\lambda)$ and $B(b,\lambda)$ of measurements on particles A and B with ${}_{AB}\!<\psi|(\sigma_A \cdot a)|\psi>_{AB}(\lambda)$ and ${}_{AB}\!<\psi|(\sigma_B \cdot b)|\psi>_{AB}(\lambda)$, respectively. The discrepancy between (2) and (3) are obvious, especially to the entangle state. For example, in the case of singlet state, the Expression (3) in Bell's theorem is:

$$\begin{aligned} P(a,b) = &\int d\lambda \rho(\lambda) \frac{1}{2} \\ &[{}_A\!<\uparrow|(\sigma_A \cdot a)|\uparrow>_A(\lambda)\, {}_B\!<\downarrow|(\sigma_B \cdot b)|\downarrow>_B(\lambda) \\ &+ {}_A\!<\downarrow|(\sigma_A \cdot a)|\downarrow>_A(\lambda)\, {}_B\!<\downarrow|(\sigma_B \cdot b)|\downarrow>_B(\lambda) \\ &+ {}_A\!<\uparrow|(\sigma_A \cdot a)|\uparrow>_A(\lambda)\, {}_B\!<\uparrow|(\sigma_B \cdot b)|\uparrow>_B(\lambda) \\ &+ {}_A\!<\downarrow|(\sigma_A \cdot a)|\downarrow>_A(\lambda)\, {}_B\!<\uparrow|(\sigma_B \cdot b)|\uparrow>_B(\lambda)], \end{aligned} \tag{4}$$

and the Expectation Value (2) in quantum mechanics will become:

$$
\begin{aligned}
&{}_{AB}\langle\psi|(\sigma_A\cdot a)(\sigma_B\cdot b)|\psi\rangle_{AB}= \\
&\int d\lambda\rho(\lambda)\frac{1}{2}[{}_A\langle\uparrow|(\sigma_A\cdot a)|\uparrow\rangle_A(\lambda)\,{}_B\langle\downarrow|(\sigma_B\cdot b)|\downarrow\rangle_B(\lambda) \\
&-{}_A\langle\uparrow|(\sigma_A\cdot a)|\downarrow\rangle_A(\lambda)\,{}_B\langle\downarrow|(\sigma_B\cdot b)|\uparrow\rangle_B(\lambda) \\
&-{}_A\langle\downarrow|(\sigma_A\cdot a)|\uparrow\rangle_A(\lambda)\,{}_B\langle\uparrow|(\sigma_B\cdot b)|\downarrow\rangle_B(\lambda) \\
&+{}_A\langle\downarrow|(\sigma_A\cdot a)|\downarrow\rangle_A(\lambda)\,{}_B\langle\uparrow|(\sigma_B\cdot b)|\uparrow\rangle_B(\lambda)],
\end{aligned} \tag{5}
$$

which clearly shows the differences of the expectation values in Bell's theorem and quantum mechanics. These differences originate from the correlation between particles A and B.

As a matter of fact, the expectation value in quantum mechanics contains the correlation between Particles A and B, while the expectation value in Bell's theorem is the product of outcomes of measurements on Particles A and B, the expectation value $A(a,\lambda)$ of Particle A is independent of the expectation value of Particle B and conversely, these independent outcomes $A(a,\lambda)$ and $B(b,\lambda)$ of measurements have destroyed the correlation between Particles A and B, and there are no correlation in the expectation value $P(a,b)$ of Bell's theorem, which leads to the differences shown in the above. However, if we chose a non-entangle state $|\psi\rangle_{AB}=|\uparrow\rangle_A|\downarrow\rangle_B$, the differences between the two expectation values will vanish, because there are no correlation between A and B in the non-entangle state.

It is this difference between the two expectation values that leads to the dissatisfaction of expectation value in quantum mechanics with Bell's inequality. In the usual reasoning of Bell's inequality, the quantity $\int d\lambda\rho(\lambda)[A(a,\lambda)B(b,\lambda)-A(a,\lambda)B(b',\lambda)]$ is naturally divided as

$$\int d\lambda\rho(\lambda)\{[A(a,\lambda)B(b,\lambda)[1\pm A(a',\lambda)B(b',\lambda)]\}-\int d\lambda\rho(\lambda)\{A(a,\lambda)B(b',\lambda)[1\pm A(a',\lambda)B(b,\lambda)]\}]$$

where a', b' are the other vectors in three -space. This rearrangement is reasonable in Bell's theorem because there are no correlation between $A(a,\lambda)$ and $B(b,\lambda)$, while errors will occur in quantum mechanics, in which we can not write out the corresponding expression, i.e.

$$
\begin{aligned}
&{}_{AB}\langle\psi|(\sigma_A\cdot a)(\sigma_B\cdot b)-(\sigma_A\cdot a)(\sigma_B\cdot b')|\psi\rangle_{AB}= \\
&{}_{AB}\langle\psi|(\sigma_A\cdot a)(\sigma_B\cdot b)|\psi\rangle_{AB} \\
&[1\pm{}_{AB}\langle\psi|(\sigma_A\cdot a')(\sigma_B\cdot b')|\psi\rangle_{AB} \\
&-{}_{AB}\langle\psi|(\sigma_A\cdot a)(\sigma_B\cdot b')|\psi\rangle_{AB} \\
&[1\pm{}_{AB}\langle\psi|(\sigma_A\cdot a')(\sigma_B\cdot b)|\psi\rangle_{AB},
\end{aligned} \tag{6}
$$

which is wrong. We can check it by a simple example. Choosing the angle between Vectors 'a' and 'b', 'b' and 'c' are 60^o in order and 'a' and 'b' as the same Vector 'c'. According to quantum mechanics, the left side of expression (6) is $-\frac{1}{2}-\frac{1}{2}=-1$, while the right side is $-\frac{1}{2}[1\pm(-1)]-\frac{1}{2}[1\pm(-\frac{1}{2})]$, they are not equal. The above rearrangement is violated in quantum mechanics, and the derivation of Bell's inequality can not be resumed considering the correlation in quantum mechanics.

So far, we conclude that the expectation value in quantum mechanics doesn't meet Bell's inequality, there exists difference between the expectation value using the probability distribution in Bell's inequality and the expectation value in quantum mechanics, the latter contain the correlations between particles, while the former not. Our analyses have some similarities with Jaynes[7]. Jaynes' main contention was that Bell's factorization for the probability of joint outcomes A and B of the two measurements does not follow the rules of probability theory. Bell's factorization is

$$P(A,B|a,b,c,\lambda)=P(A|a,c,\lambda)P(B|b,c,\lambda),$$

while the correct factorization should be $P(A,B|a,b,c,\lambda)=P(A|B,a,b,c,\lambda)P(B|a,b,c,\lambda)$, which is analogous to our analysis. In our treatment, the expectation value containing the correlation between Particles A and B can not be divided into the product of two single expectation values for Particles A and B, that means

$$
\begin{aligned}
&{}_{AB}\langle\psi|(\sigma_A\cdot a)(\sigma_B\cdot b)|\psi\rangle_{AB}\neq \\
&{}_{AB}\langle\psi|(\sigma_A\cdot a)|\psi\rangle_{AB}(\lambda)\,{}_{AB}\langle\psi|(\sigma_B\cdot b)|\psi\rangle_{AB}(\lambda)
\end{aligned}
$$

Although the expectation value

$${}_{AB}\langle\psi|(\sigma_A\cdot a)(\sigma_B\cdot b)|\psi\rangle_{AB}$$

here is different from the probability in Jaynes' reasoning and can't be factorized into the product of two terms as Jaynes', there still exist similarities between them.

In fact, the expectation values in quantum mechanics obey their own inequality. As we know, $\rho(\lambda)$ is the probability distribution of local hidden-variable λ in Bell's theorem. However, if we interpret it as the density matrix of quantum state, which is different from the original definition of $\rho(\lambda)$, we can define the expectation value in Bell's inequality as

$$P(a,b)=tr[\int d\quad(\lambda)(\sigma_A\cdot a)(\sigma_B\cdot b)], \tag{7}$$

which has the similar form with Bell's Definition (1). But here it is identical to the quantum mechanics Expectation

Value (2), and the difference of Expectation Values between Bell's inequality and quantum mechanics will vanish. This generalized Expectation Value (7) satisfies

$$-1 \le tr[\int d\lambda\rho(\lambda)(\sigma_A \cdot a)(\sigma_B \cdot b)] \le 1,$$

so we have

$$|tr[\int d\lambda\rho(\lambda)(\sigma_A \cdot a)(\sigma_B \cdot b)]$$
$$-tr[\int d\lambda\rho(\lambda)(\sigma_A \cdot a)(\sigma_B \cdot c)]|$$
$$+|tr[\int d\lambda\rho(\lambda)(\sigma_A \cdot b)(\sigma_B \cdot c)]| \le 3,$$

which can be further rearranged as:

$$|P(a,b)-P(a,c)| \le 3-|P(b,c)|, \qquad (9)$$

so that we have obtained the inequality satisfied by the expectation value in quantum mechanics, which has similar form with the original Bell's inequality. Inequality (9) has contained the correlation between Particles A and B, and must coincide with the prediction of quantum mechanics.

III. CONCLUSIONS

In Aspect et al.'s experiment, the expectation value ${}_{AB}\!<\psi|(\sigma_A \cdot a)(\sigma_B \cdot b)|\psi>_{AB}$ involving the correlation of two photons is measured by the time-varying analyzers, which is not the same quantity $P(a,b)$ in Bell's inequality, but the expectation value in quantum mechanics. So it violates the Bell's inequality and is coincided with quantum mechanics. However, the expectation value measured in experiment should satisfy the Inequality (9) presented by us. Inequality (9) may be tested by other experiments, too.

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