Mathematical Modeling of Tall Buildings and Its Foundation under Randomly Fluctuating Wind and Earthquake Ground Motions

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***Abstract*-In the present paper, a non-dimensional mathematical model for high tower buildings and its foundation under randomly fluctuating wind loads and earthquake ground motions excitations is developed as a nonlinear model to study the system more extensively. The main equations of the system could be derived using two different derivation methods, linearized in minimal symbolic forms; which facilitate a subsequent numerical simulation in order to investigate the vibration characteristics of the whole system. The analysis enables designers to have more insight into the characteristics of high tower buildings of similar configuration but with different geometry and material. The complexity of wind loading with its variations in space and time has been considered. A comprehensive mathematical model of six degrees of freedom is presented and solved for free and forced vibrations.**

Keywords- Tall Building Vibrations; Modal Analysis; Foundation Vibrations; Power Spectral Density; Random Wind Excitation; Earthquake Ground Motions

I. INTRODUCTION AND LITERATURE REVIEW

Large investments have recently been made for the construction of new medium- and high-rise buildings in the world. In many cases performance-based designs have been the preferred method for these buildings. A main consideration in performance-based seismic design is the estimation of the likely development of structural and nonstructural damage limit-states is given a hazard level. For this type of buildings efficient modeling techniques are required able to compute the response at different performance states. Certain structures are less vulnerable against vibration impacts whereas certain others are more vulnerable. As we all know that vibration effects are now cannot be neglected, as our day to day life is affected by them. Study of vibration responses of structures has always been a principal concern for design engineers. Therefore, we do put an eye on the vibrations of buildings and its foundations. Uncontrolled vibration causes devastation. Occurrences of Tsunami, earthquake, collapse of structures are few such most common devastating effects of vibration. Thus the study of vibration responses in advance is of immense importance for sustainable and positive effects of vibrations for the well being of humans.

Nowadays, the new and emerging concept of seismic structural design, the so-called performance-based design, requires careful consideration of all aspects involved in structural analysis. One of the most important aspects of structural analysis is Soil-Structure Interaction (SSI). Such interaction may alter the dynamic characteristics of structures and consequently may be beneficial or detrimental to the performance of structures. Soil conditions at a given site may amplify the response of a structure on a soil deposit. Not taking into account these structural response amplifications may lead to an under-designed structure resulting in a premature collapse during an earthquake. Analytical methods of SSI concentrate mainly on single degree of freedom systems and analysis/design of long and important structures such as large bridges and nuclear power plants, and rarely on regular type buildings. Studies which include SSI effects will help to better predict the performance of structures during future ground motions. State of the art knowledge and analytical approaches require that, the structure-foundation system to be represented by mathematical models includes the influence of the sub-foundation media.

A research work of Panagiotou, M. (2008) was conducted at University of California San Diego (UCSD) on the seismic design, experimental response, and computational modeling of medium- and high-rise reinforced concrete wall buildings. Kim, S.J. (2008) presented an investigation of the effect of vertical ground motion on reinforced concrete structures studied through a combined analytical-experimental research approach. Krier, D. (2009) analyzed several soil-structure interaction problems. Buildings on elastic foundations were studied and comparisons were made between analytical results and the solutions obtained from a Tera Dysac finite element analysis. Gouasmia, A. et al. (2009) studied the seismic response of an idealized small city composed of five equally spaced, five storey reinforced concrete buildings anchored in a soft soil layer overlaid by a rock half space. These results show response amplification of the buildings in the near field in accordance with the results observed in similar cases. Antonyuk, E.Ya., Timokhin, V.V. (2007) outlined a mathematical model describing the vibrations of buildings and engineering structures with general-type passive shock-absorbers, rigid bodies, and ideal constraints.

Auersch, L. (2008) predicted a practice-oriented environmental building vibration. A Green's functions method for layered soils is used to build the dynamic stiffness matrix of the soil area that is covered by the foundation. A simple building model is proposed by adding a building mass to the dynamic stiffness of the soil. Belakroum, R., et al. (2008) studied the numerical prediction of the aerodynamic behavior of rectangular buildings. Simulations were made for rectangles of different side coefficients and different angles of attack. The finite element method is used to simulate fluid flow considered Newtonian and incompressible. Davoodi, M., et al. (2008) used the ambient vibration tests to rely on natural excitations; consequently, it was recommended to perform impulsive test for identifying the hidden dynamic characteristics of the building. Kuźniar, K. and Waszczyszyn, Z. (2006) applied neural networks for computation of fundamental natural periods of buildings with load-bearing walls. The analysis is based on long-term tests performed on actual buildings. The identification problem was formulated as the relation between structural and soil basement parameters, and the fundamental period of building.

Uzdin, A.M. et al. (2009) derived equations for the vibrations of a building on the foundations under consideration. Impossibility of use of traditional methods of the linear-spectral theory for analysis of their earthquake resistance is demonstrated. It is established that the systems under consideration do not possess a natural vibration period, and may have ambiguous solutions for forced vibrations. The influence of city traffic-induced vibration on Vilnius Arch-Cathedral Belfry was investigated (Kliukas, R. et al. 2008). Two sources of dynamic excitation were studied. Conventional city traffic was considered to be a natural source of excitation while excitation imposed artificially by moving a heavily loaded truck was considered to be the source of increased risk excitation. Configuration of equipment on springs is simplified for numerical analysis. A simplified approach and associated equations of motion can be developed to evaluate the response of the equipment with vertical and horizontal forcing functions (Turner, J. 2004). Gong, Y. (2010) developed a free vibration analysis method for space mega frames of super tall buildings. The physical model of a mega frame was idealized as a three-dimensional assemblage of stiffened close-thin-walled tubes with continuously distributed mass and stiffness.

Yang, Y.B. et. al analyzed the wave propagation problems caused by the underground moving trains by the 2.5-dimensional finite/infinite element approach. The near field of the half-space, including the tunnel and parts of the soil, was simulated by finite elements, and the far field extending to infinity by infinite elements. Ground-borne vibrations due to subway trains have sometimes reached the level that cannot be tolerated by residents living in adjacent buildings (Shyu et. al. 2002). Also, approaches for predicting vibrations caused by metro trains moving through the tunnel were developed (Gupta et al. 2007), e.g., a semi-analytical pipe-in-pipe model (Forrest and Hunt 2006a, b) and a coupled periodic finite-element–boundary-element model (Clouteau et al. 2005; Degrande et al. 2006b). Clearly, ground-borne vibrations have become an issue of great concern, which will continuously attract the attention of researchers and engineers worldwide. Many research projects on ground-borne vibrations due to subway trains were conducted by field measurement (Vadillo et al. 1996; Degrande et al. 2006a) and empirical or semiempirical prediction models (Kurzweil 1979; Trochides 1991; Melke 1998). These studies provide practical references for solving related problems. However, most of these studies were performed for a specific condition, thereby suffering from the lack of generality. On the other hand, concerning the techniques of simulation, most previous works have been based on the two-dimensional (2D) models (Balendra et al. 1991; Yun et al. 2000; Metrikine and Vrouwenvelder 2000).

Prowell, I. (2011) presented an experimental and numerical investigation into the seismic response of modern wind turbines simultaneously subjected to wind, earthquake, and operational excitation. Ulusoy, H.S. (2011) described a certain class of system identification algorithms with particular emphasis on civil engineering applications. The algorithms originated from system realization theory enabled one to identify finite dimensional, linear, time-invariant models of systems in the state space representation from observed data. Wieser, J. (2011) used OpenSees finite element framework to develop full three dimensional models of four steel moment frame buildings. The incremental dynamic analysis method is employed to evaluate the floor response of inelastic steel moment frame buildings subjected to all three components of a suite of 21 ground motions. Ghafari Oskoei, S.A. (2011) dealt with the dynamic behavior of tall guyed masts under seismic loads. Zhong, P. (2011) utilized a ground motion acceleration time-history as an input to an analytic model of a structure and solved the structural response at each time steps of the ground motion record.

Weng, S. (2010) proposed a forward sub-structuring approach, the eigen-properties of the partitioned substructures were assembled to recover the eigen-solutions and eigen-sensitivities of the global structure, which were tuned to reproduce the experimental measurements through an optimization process. Sonmez, E. (2010) developed semi-active controllers, which were based on real-time estimation of instantaneous (dominant) frequency and the evolutionary power spectral density by time-frequency analysis of either the excitation or the response of the structure. Time-frequency analyses were performed by either short-time Fourier transform or wavelet transform. Soudkhah, M. (2010) examined the dynamic response of surface foundations on sandy soils under both forced and ground motion disturbance. Yao, M.M. (2010) used the direct method for modeling the soil and a tall building together and studied energy transferring from soils to buildings during earthquakes, which is critical for the design of earthquake resistant structures and for upgrading existing structures. Ahearn, E.B. (2010) studied the dynamic effects of wind-induced vibrations on high-mast structures in Laramie, WY, and proposed several retrofits that increase the aerodynamic damping, thereby reducing vibrations.

The ground vibration induced by earthquake ground motions is a complicated dynamic problem due to the involvement of a number of factors along the paths of wave propagation, including the load generation mechanism, the geometry and location of tunnel structures, the irregularity of soil layers, etc. Previously, many research projects on ground-borne vibrations due to earthquakes were conducted by field measurement and empirical or semi-empirical prediction models. These studies provide practical reference for solving related problems. However, most of these studies were performed for a specific condition, thereby suffering from the lack of generality.

II. ASSUMPTIONS

1. The high tower building-foundation equivalent system moves only in the y^*- z^* plane.
2. The wind effect is identified as randomly fluctuating wind loads in horizontal direction.
3. $U_y(t)$, $U_Z(t)$ are random ground motions of earthquakes in horizontal y and vertical directions z .
4. The high tower building and its foundation are assumed as rigid bodies.
5. The soil kind under the foundation is assumed as sandy clay.

6. The angular velocities $\varphi_o^*(t), \varphi_1^*(t)$, and $\varphi_2^*(t)$ are very small (<<1).
7. The equivalent spring stiffness k_H, k_{EH}, and k_v are linear.
8. The equivalent damping coefficients r_H, r_{EH}, and r_v are linear.
9. The density of building ρ_2 is taken as 0.1 that of the foundation.
10. The air friction was not considered.
11. The place pressure factor C_p can be replaced through the average load factor of total building.
12. The spectral power density $S_{U_1U_1}(\Omega)$ is independent on the Cartesian Coordinates z, y.
13. The wind velocity distribution along the height of the building is $\overline{U}(z) = (\frac{z}{H})^{\alpha}\overline{U}(H)$.
14. The cross spectral power density $S_{U_1U_2}(\Omega)$ can be represented through the coherence spectrum of the wind velocity $U'(z_1,t)$ and $U'(z_2,t)$:

$$\gamma^2_{U_1U_2}(\Omega) = \left|S_{U_1U_2}(\Omega)\right|^2 \Big/ [S_{U_1U_1}(\Omega).S_{U_2U_2}(\Omega)]$$

III. Derivation of System Equations Using D'Alembert's Principle

The model of the problem to be considered is schematically shown in Fig. 1. This model describes the vibrations of high-tower building and its foundation with general-type equivalent passive springs and dampers, rigid bodies, and some ideal constraints like linear springs and dampers under the effect of randomly fluctuating wind loads and the excitation of earthquake ground motions. In setting up the equations of motion of the equivalent system in Fig. 1, it should be born in mind that the geometric, elastic, and kinetic relations of both high tower building and its foundation must be derived. Moreover, the external excitation of wind loads should be prepared.

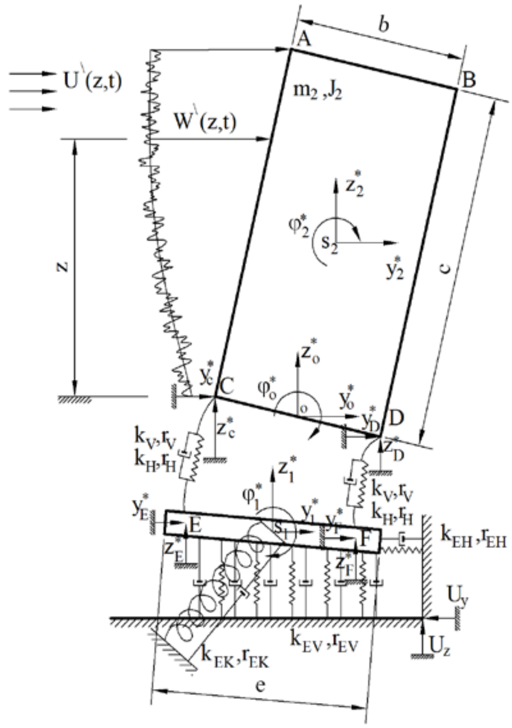

Fig. 1 Equivalent system of tall building and its foundation

A. Foundation Differential Equations of Motion

Fig. 2 shows the free body diagram of foundation with its accompanied vibrating soil.

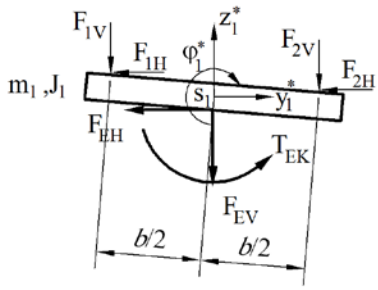

Fig. 2 Free body diagram of foundation with its accompanied vibrated soil

B. Geometric Relations of Tall Building and Its Foundation

For the linearization of derived equations, let φ_o, φ_1 and φ_2 << 1. Geometric relations of building's foundation are

$$z_C^*(t) = z_o^*(t) + 0.5b.\varphi_o^*(t),\ z_D^*(t) = z_o^*(t) - 0.5b.\varphi_o^*(t),\ z_E^*(t) = z_1^*(t) + 0.5b.\varphi_1^*(t),\ z_F^*(t) = z_1^*(t) - 0.5b.\varphi_1^*(t),$$

$$z_2^*(t) = z_o^*(t) - 0.5c.(1 - \cos\varphi_2^*(t)) \approx z_o^*(t),\ \varphi_2^*(t) = \varphi_o^*(t),\ y_C^*(t) = y_o^*(t),$$

$$y_2^*(t) = y_o^*(t) + 0.5c.\sin\varphi_2^*(t) \approx y_o^*(t) + 0.5c.\varphi_2^*(t),\ y_D^*(t) = y_o^*(t),\ y_E^*(t) = y_1^*(t), \text{ and } y_F^*(t) = y_1^*(t)$$

Rearranging the previous geometric relations leads to the following form

$$\left.\begin{array}{l} z_C^*(t) = z_2^*(t) + 0.5b.\varphi_2^*(t),\ z_D^*(t) = z_2^*(t) - 0.5b.\varphi_2^*(t),\ z_E^*(t) = z_1^*(t) + 0.5b.\varphi_1^*(t),\ z_F^*(t) = z_1^*(t) - 0.5b.\varphi_1^*(t) \\ y_C^*(t) = y_2^*(t) - 0.5c.\varphi_2^*(t),\ y_D^*(t) = y_2^*(t) - 0.5c.\varphi_2^*(t),\ y_E^*(t) = y_1^*(t), \text{ and } y_F^*(t) = y_1^*(t) \end{array}\right\} \quad (1)$$

C. Elastic Relations of Building's Foundation

Elastic relations of building's foundation have the form

$$F_{1V} = k_V.[z_E^*(t) - z_C^*(t)] + r_V.[\dot{z}_E^*(t) - \dot{z}_C^*(t)],\ F_{1H} = k_H.[y_E^*(t) - y_C^*(t)] + r_H.[\dot{y}_E^*(t) - \dot{y}_C^*(t)],$$

$$F_{2V}=k_V.[z_F^*(t)-z_D^*(t)]+r_V.[\dot z_F^*(t)-\dot z_D^*(t)]\,,\ F_{2H}=k_H.[y_F^*(t)-y_D^*(t)]+r_H.[\dot y_F^*(t)-\dot y_D^*(t)]\,,$$

$$\left.\begin{array}{l} F_{EH}=k_{EH}.[y_1^*(t)-U_y(t)]+r_{EH}.[\dot y_1^*(t)-\dot U_y(t)],\ F_{EV}=k_{EV}.[z_1^*(t)-U_z(t)]+r_{EV}.[\dot z_1^*(t)-\dot U_z(t)] \end{array}\right\} \tag{2}$$

$$T_{EK}=k_{EK}.\varphi_1^*(t)+r_{EK}.\dot\varphi_1^*(t)$$

D. Kinetic Relations of Building's Foundation

Applying Newton's second law for the forces in z- and y-directions and the moments about s_1 results in

$$\sum F_z=m_1.\ddot z_1^*(t)=-F_{1V}-F_{2V}-F_{EV}\,,$$
$$\sum F_y=m_1.\ddot y_1^*(t)=-F_{1H}-F_{2H}-F_{EH}$$
$$\sum M_{s1}=J_1.\ddot\varphi_1^*(t)=-F_{1V}.0.5b.\cos\varphi_1^*(t)$$
$$+F_{2V}.0.5b.\cos\varphi_1^*(t)-T_{EK}(t)$$
$$\approx(F_{2V}-F_{1V}).0.5b-T_{EK}(t) \tag{3}$$

E. Differential Equations of Motion of High Tower Building

Fig. 3 shows the free body diagram of high tower building with its forces and moments affecting on it.

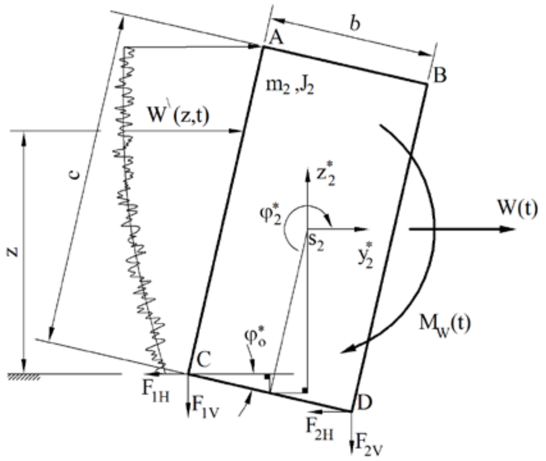

Fig. 3 Free body diagram of the high tower building

F. Aeroelastic Relations of Wind Excitation

Nowadays, the study of the behavior of a structure subjected to hydro or aerodynamic loadings forms an integral part of tasks allocated to engineers. The effect of wind must be taken into consideration during the design phase of tall buildings. The mechanism of wind loads acting on a building is very complex. Substantial works have dealt with this problem. In civil engineering and construction of tall buildings, the assessment of wind loads is required to check the resistance of components of the construction and coating. In recent years, the methods proposed by scientists in this field are constantly being updated. The institutions of global standardization are thus forced each time to review the standards that are in force. Under the effects of wind, a building oscillates according to both directions parallel and perpendicular to the flow and in a torsional mode. Notwithstanding its enormous fascination, wind loading is in fact a parasitic effect, and mostly an obstacle in the way of designing structures for their primary intended use. Without wind, structures – particularly large ones – would probably be a lot easier to design and cheaper.

Dynamic wind pressures acting on buildings are complicated functions of both time and space. The wind load per unit area has the form

$$W(z,t)=C_p.q(z,t)\quad\text{and}\quad q(z,t)=\frac{1}{2}\rho\,U^2(z,t)$$

$$W(z,t)=C_p.\frac{\rho}{2}.U^2(z,t)=C_p.\frac{\rho}{2}.[\overline{U}(z)+U'(z,t)]^2=$$

$$C_p.\frac{\rho}{2}.[\overline{U}^2(z)+2\overline{U}(z).U'(z,t)+U'^2(z,t)]$$

$$W(z,t)=C_p.\frac{\rho}{2}.\overline{U}^2(z)+C_p.\rho.\overline{U}(z).U'(z,t)=\overline{W}(z)+W'(z,t)$$

The total turbulent wind force in y^*-direction as a function of time is

$$W(t)=\int_0^c W'(z,t)\,dz=\int_0^c C_p.\rho.\overline{U}(z).U'(z,t)\,dz \tag{4}$$

The total turbulent wind moment as a function of time is

$$M_W(t)=\int_0^c[z-(\frac{c}{2}\cos\varphi_2^*(t)-\frac{b}{2}\sin\varphi_2^*(t))].W'(z,t)\,dz$$

$$\approx\int_0^c(z-\frac{c}{2}).W'(z,t)\,dz=\int_0^c(z-\frac{c}{2}).\,C_p.\rho.\overline{U}(z).U'(z,t)\,dz \tag{5}$$

G. Elastic Relations of High Tower Building

Elastic relations of high tower building have the form

$$F_{1V}=k_V.[z_C^*(t)-z_E^*(t)]+r_V.[\dot z_C^*(t)-\dot z_E^*(t)]\,,$$
$$F_{1H}=k_H.[y_C^*(t)-y_E^*(t)]+r_H.[\dot y_C^*(t)-\dot y_E^*(t)]$$
$$F_{2V}=k_V.[z_D^*(t)-z_F^*(t)]+r_V.[\dot z_D^*(t)-\dot z_F^*(t)]\,,$$
$$F_{2H}=k_H.[y_D^*(t)-y_F^*(t)]+r_H.[\dot y_D^*(t)-\dot y_F^*(t)] \tag{6}$$

H. Kinetic Relations of High Tower Building

Applying Newton's second law for the forces in z and y-directions and also the moments about s_2 results in

$$\sum F_z=m_2.\ddot z_2^*(t)=-F_{1V}-F_{2V}\,,$$

$$\sum F_y=m_2.\ddot y_2^*(t)=-F_{1H}-F_{2H}+W(t)$$

$$\sum M_{s2}=J_2.\ddot\varphi_2^*(t)=-F_{1V}.[\frac{c}{2}.\sin\varphi_2^*(t)+\frac{b}{2}.\cos\varphi_2^*(t)]+F_{1H}.[\frac{c}{2}.\cos\varphi_2^*(t)-\frac{b}{2}.\sin\varphi_2^*(t)]$$

$$+F_{2V}.[-\frac{c}{2}.\sin\varphi_2^*(t)+\frac{b}{2}.\cos\varphi_2^*(t)]+F_{2H}.[\frac{c}{2}.\cos\varphi_2^*(t)+\frac{b}{2}.\sin\varphi_2^*(t)]+M_W(t)\} \quad (7)$$

The previous equation can be linearized in the following form

$$\sum M_{s2}\approx -F_{1V}.[\frac{b}{2}+\frac{c}{2}.\varphi_2^*(t)]+F_{1H}.[-\frac{b}{2}.\varphi_2^*(t)+\frac{c}{2}]+F_{2V}.[\frac{b}{2}-\frac{c}{2}.\varphi_2^*(t)]+F_{2H}.[\frac{b}{2}.\varphi_2^*(t)+\frac{c}{2}]+M_W(t)$$

I. Deriving the System's Differential Equations of Motion Application of the Geometric Relations of the Foundation

Substitute from Eqs. 1 in Eqs. 2 of the elastic relations of foundation free body diagram

$$\left.\begin{aligned}
F_{1V}&=k_V.[z_1^*(t)+0.5b.\varphi_1^*(t)-z_2^*(t)-0.5b.\varphi_2^*(t)]+r_V.[\dot z_1^*(t)+0.5b.\dot\varphi_1^*(t)-\dot z_2^*(t)-0.5b.\dot\varphi_2^*(t)]\\
F_{1H}&=k_H.[y_1^*(t)-y_2^*(t)+0.5c\varphi_2^*(t)]+r_H.[\dot y_1^*(t)-\dot y_2^*(t)+0.5c\dot\varphi_2^*(t)]\\
F_{2V}&=k_V.[z_1^*(t)-0.5b.\varphi_1^*(t)-z_2^*(t)+0.5b.\varphi_2^*(t)]+r_V.[\dot z_1^*(t)-0.5b.\dot\varphi_1^*(t)-\dot z_2^*(t)+0.5b.\dot\varphi_2^*(t)]\\
F_{2H}&=k_H.[y_1^*(t)-y_2^*(t)+0.5c\varphi_2^*(t)]+r_H.[\dot y_1^*(t)-\dot y_2^*(t)+0.5c\dot\varphi_2^*(t)]
\end{aligned}\right\} \quad (8)$$

J. Application of the Elastic Relations of the Foundation

Substituting from Eqs. 2 of foundation's elastic relations in Eqs. 3 of its kinetic relations results in

$$\left.\begin{aligned}
m_1.\ddot z_1^*(t)&=-(k_{EV}+2k_V).z_1^*(t)+2k_V z_2^*(t)-(r_{EV}+2r_V).\dot z_1^*(t)+2r_V.\dot z_2^*(t)+k_{EV}.U_z(t)+r_{EV}\dot U_z(t)\\
m_1.\ddot y_1^*(t)&=-(k_{EH}+2k_H).y_1^*(t)+2k_H.y_2^*(t)-ck_H.\varphi_2^*(t)-(r_{EH}+2r_H).\dot y_1^*(t)\\
&\quad+2r_H.\dot y_2^*(t)-cr_H.\dot\varphi_2^*(t)+k_{EH}.U_y(t)+r_{EH}.\dot U_y(t)\\
J_1.\ddot\varphi_1^*(t)&=-[k_{EK}+0.5b^2.k_V].\varphi_1^*(t)+0.5b^2.k_V.\varphi_2^*(t)-[r_{EK}+0.5b^2.r_V].\dot\varphi_1^*(t)+0.5b^2.r_V.\dot\varphi_2^*(t)
\end{aligned}\right\} \quad (9)$$

K. Application of the Geometric Relations of the Building

Substituting from Eqs. 1 of geometric relations in Eqs. 6 of elastic relations of the building

$$\left.\begin{aligned}
F_{1V}&=k_V.[z_2^*(t)+0.5b.\varphi_2^*(t)-z_1^*(t)-0.5b.\varphi_1^*(t)]+r_V.[\dot z_2^*(t)+0.5b.\dot\varphi_2^*(t)-\dot z_1^*(t)-0.5b.\dot\varphi_1^*(t)]\\
F_{1H}&=-k_H.[y_1^*(t)-y_2^*(t)+0.5c\varphi_2^*(t)]-r_H.[\dot y_1^*(t)-\dot y_2^*(t)+0.5c\dot\varphi_2^*(t)]\\
F_{2V}&=k_V.[z_2^*(t)-0.5b.\varphi_2^*(t)-z_1^*(t)+0.5b.\varphi_1^*(t)]+r_V.[\dot z_2^*(t)-0.5b.\dot\varphi_2^*(t)-\dot z_1^*(t)+0.5b.\dot\varphi_1^*(t)]\\
F_{2H}&=-k_H.[y_1^*(t)-y_2^*(t)+0.5c\varphi_2^*(t)]-r_H.[\dot y_1^*(t)-\dot y_2^*(t)+0.5c\dot\varphi_2^*(t)]
\end{aligned}\right\} \quad (10)$$

L. Application of the Elastic Relations of the Building

Substituting Eqs. 10 of building's elastic relations in Eqs. 7 of its kinetic relations lead to the following differential equations

$$\left.\begin{aligned}
m_2.\ddot z_2^*(t)&=2k_V.z_1^*(t)-2k_V.z_2^*(t)+2r_V.\dot z_1^*(t)-2r_V.\dot z_2^*(t)\\
m_2.\ddot y_2^*(t)&=2k_H.y_1^*(t)-2k_H.y_2^*(t)+ck_H.\varphi_2^*(t)+2r_H.\dot y_1^*(t)-2r_H.\dot y_2^*(t)+cr_H.\dot\varphi_2^*(t)]+W(t)\\
J_2.\ddot\varphi_2^*(t)&=-ck_H.y_1^*(t)+0.5b^2.k_V.\varphi_1^*(t)+ck_H.y_2^*(t)-(0.5c^2.k_H+0.5b^2.k_V).\varphi_2^*(t)\\
&\quad-cr_H.\dot y_1^*(t)+0.5b^2.r_V.\dot\varphi_1^*(t)+cr_H.\dot y_2^*(t)-(0.5c^2.r_H+0.5b^2.r_V).\dot\varphi_2^*(t)+M_W(t)
\end{aligned}\right\} \quad (11)$$

M. Arranging the Differential Equations of Motion

The differential equations of motion of both tall building and its foundation can be summarized in the form

$$m_1.\ddot z_1^*(t)+(r_{EV}+2r_V).\dot z_1^*(t)-2r_V.\dot z_2^*(t)+(k_{EV}+2k_V).z_1^*(t)-2k_V z_2^*(t)=k_{EV}.U_z(t)+r_{EV}\dot U_z(t)$$

$$m_1.\ddot y_1^*(t)+(r_{EH}+2r_H).\dot y_1^*(t)-2r_H.\dot y_2^*(t)+cr_H.\dot\varphi_2^*(t)+(k_{EH}+2k_H).y_1^*(t)-2k_H.y_2^*(t)+ck_H.\varphi_2^*(t)=$$
$$k_{EH}U_y(t)+r_{EH}.\dot U_y(t)$$

$$J_1.\ddot\varphi_1^*(t)+[r_{EK}+0.5b^2.r_V].\dot\varphi_1^*(t)-0.5b^2.r_V.\dot\varphi_2^*(t)+(k_{EK}+0.5b^2.k_V).\varphi_1^*(t)-0.5b^2.k_V.\varphi_2^*(t)=0$$

$$m_2.\ddot z_2^*(t)-2r_V.\dot z_1^*(t)+2r_V.\dot z_2^*(t)-2k_V.z_1^*(t)+2k_V.z_2^*(t)=0$$

$$m_2.\ddot{y}_2^*(t) - 2r_H.\dot{y}_1^*(t) + 2r_H.\dot{y}_2^*(t) - cr_H.\dot{\varphi}_2^*(t) - 2k_H.y_1^*(t) + 2k_H.y_2^*(t) - ck_H.\varphi_2^*(t) = W(t)$$

$$J_2.\ddot{\varphi}_2^*(t) + cr_H.\dot{y}_1^*(t) - 0.5b^2.r_V.\dot{\varphi}_1^*(t) - cr_H.\dot{y}_2^*(t) + (0.5c^2.r_H + 0.5b^2.r_V).\dot{\varphi}_2^*(t)$$

$$+ ck_H.y_1^*(t) - 0.5b^2.k_V.\varphi_1^*(t) - ck_H.y_2^*(t) + (0.5c^2.k_H + 0.5b^2.k_V).\varphi_2^*(t) = M_W(t)$$

$$\begin{bmatrix} m_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & J_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & J_2 \end{bmatrix} \begin{bmatrix} \ddot{z}_1^*(t) \\ \ddot{y}_1^*(t) \\ \ddot{\varphi}_1^*(t) \\ \ddot{z}_2^*(t) \\ \ddot{y}_2^*(t) \\ \ddot{\varphi}_2^*(t) \end{bmatrix} + \begin{bmatrix} r_{EV}+2r_V & 0 & 0 & -2r_V & 0 & 0 \\ 0 & r_{EH}+2r_H & 0 & 0 & -2r_H & cr_H \\ 0 & 0 & r_{EK}+0.5b^2 r_V & 0 & 0 & -0.5b^2 r_V \\ -2r_V & 0 & 0 & 2r_V & 0 & 0 \\ 0 & -2r_H & 0 & 0 & 2r_H & -cr_H \\ 0 & cr_H & -0.5b^2 r_V & 0 & -cr_H & (0.5c^2 r_H + 0.5b^2 r_V) \end{bmatrix} \begin{bmatrix} \dot{z}_1^*(t) \\ \dot{y}_1^*(t) \\ \dot{\varphi}_1^*(t) \\ \dot{z}_2^*(t) \\ \dot{y}_2^*(t) \\ \dot{\varphi}_2^*(t) \end{bmatrix} +$$

$$\begin{bmatrix} k_{EV}+2k_V & 0 & 0 & -2k_V & 0 & 0 \\ 0 & k_{EH}+2k_H & 0 & 0 & -2k_H & ck_H \\ 0 & 0 & k_{EK}+0.5b^2 k_V & 0 & 0 & -0.5b^2 k_V \\ -2k_V & 0 & 0 & 2k_V & 0 & 0 \\ 0 & -2k_H & 0 & 0 & 2k_H & -ck_H \\ 0 & ck_H & -0.5b^2 k_V & 0 & -ck_H & (0.5c^2 k_H + 0.5b^2 k_V) \end{bmatrix} \begin{bmatrix} z_1^*(t) \\ y_1^*(t) \\ \varphi_1^*(t) \\ z_2^*(t) \\ y_2^*(t) \\ \varphi_2^*(t) \end{bmatrix} =$$

$$\begin{bmatrix} k_{EV} & r_{EV} & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{EH} & r_{EH} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi(t) \\ \dot{\xi}(t) \\ \eta(t) \\ \dot{\eta}(t) \\ W(t) \\ M_W(t) \end{bmatrix} \tag{12}$$

IV. DERIVATION OF SYSTEM EQUATIONS USING LAGRANGE'S METHOD

The previous obtained system differential Equations (12) of motion can be verified using another derivation method, like Lagrange's method using the following Lagrangian Differential Equation

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_K}\right] - \frac{\partial L}{\partial q_K} + \frac{\partial \Re}{\partial \dot{q}_K} = Q_K = \sum_i F_i \frac{\partial \nu_i}{\partial \dot{q}_K} \quad , \quad L = E - U \quad , \quad \Re = \sum_n \frac{1}{2} r_n \nu_n^2 \tag{13}$$

Q_K : General forces, F_i : External forces, and ν_i : Velocity

A. Lagrangian Function

1) Kinetic Energy of the Total Equivalent System:

$$E = \frac{1}{2}m_1\dot{z}_1^{*2}(t) + \frac{1}{2}m_1\dot{y}_1^{*2}(t) + \frac{1}{2}J_1\dot{\varphi}_1^{*2}(t) + \frac{1}{2}m_2\dot{z}_2^{*2}(t) + \frac{1}{2}m_2\dot{y}_2^{*2}(t) + \frac{1}{2}J_2\dot{\varphi}_2^{*2}(t) \tag{14}$$

2) Elastic Potential energy of the Total Equivalent System:

$$U = \frac{1}{2}k_{EV}[z_1^*(t) - U_z(t)]^2 + \frac{1}{2}k_{EH}[y_1^*(t) - U_y(t)]^2 + \frac{1}{2}k_{EK}\varphi_1^{*2}(t) + \frac{1}{2}k_V[z_E^*(t) - z_C^*(t)]^2$$

$$+ \frac{1}{2}k_H[y_E^*(t) - y_C^*(t)]^2 + \frac{1}{2}k_V[z_F^*(t) - z_D^*(t)]^2 + \frac{1}{2}k_H[y_F^*(t) - y_D^*(t)]^2 \tag{15}$$

3) Lagrangian Function:

Using Eqs. 1 and 14-15 to obtain the following Lagrangian function

$$L = \frac{1}{2}m_1\dot{z}_1^{*2}(t) + \frac{1}{2}m_1\dot{y}_1^{*2}(t) + \frac{1}{2}J_1\dot{\varphi}_1^{*2}(t) + \frac{1}{2}m_2\dot{z}_2^{*2}(t) + \frac{1}{2}m_2\dot{y}_2^{*2}(t) + \frac{1}{2}J_2\dot{\varphi}_2^{*2}(t)$$

$$- \frac{1}{2}k_{EV}[z_1^*(t) - U_z(t)]^2 - \frac{1}{2}k_{EH}[y_1^*(t) - U_y(t)]^2 - \frac{1}{2}k_{EK}\varphi_1^{*2}(t)$$

$$-\frac{1}{2}k_V[z_1^*(t)+0.5b.\varphi_1^*(t)-z_2^*(t)-0.5b.\varphi_2^*(t)]^2-\frac{1}{2}k_H[y_1^*(t)-y_2^*(t)+0.5c.\varphi_2^*(t)]^2$$

$$-\frac{1}{2}k_V[z_1^*(t)-0.5b.\varphi_1^*(t)-z_2^*(t)+0.5b.\varphi_2^*(t)]^2-\frac{1}{2}k_H[y_1^*(t)-y_2^*(t)+0.5c.\varphi_2^*(t)]^2 \quad (16)$$

B. Rayleigh's Dissipation Function

The Rayleigh's dissipation function can be derived as

$$\Re=\sum_{n=1-6}\frac{1}{2}r_6\nu_6^2=\frac{1}{2}r_{EV}[\dot{z}_1^*(t)-\dot{U}_z(t)]^2+\frac{1}{2}r_{EH}[\dot{y}_1^*(t)-\dot{U}_y(t)]^2+\frac{1}{2}r_{EK}\dot{\varphi}_1^{*2}(t)$$

$$+\frac{1}{2}r_V[\dot{z}_1^*(t)+0.5b.\dot{\varphi}_1^*(t)-\dot{z}_2^*(t)-0.5b.\dot{\varphi}_2^*(t)]^2+\frac{1}{2}r_H[\dot{y}_1^*(t)-\dot{y}_2^*(t)+0.5c.\dot{\varphi}_2^*(t)]^2$$

$$+\frac{1}{2}r_V[\dot{z}_1^*(t)-0.5b.\dot{\varphi}_1^*(t)-\dot{z}_2^*(t)+0.5b.\dot{\varphi}_2^*(t)]^2+\frac{1}{2}r_H[\dot{y}_1^*(t)-\dot{y}_2^*(t)+0.5c.\dot{\varphi}_2^*(t)]^2 \quad (17)$$

C. General External Forces $Q_K=\sum_i \underline{F}_i\frac{\partial\nu_i}{\partial\dot{q}_K}=\sum_i \underline{F}_i\frac{\partial\underline{r}_i}{\partial q_K}$, Where $F_1=W$, $F_2=M_W$, $\nu_1=\dot{y}_2^*$, and $\nu_2=\dot{\varphi}_2^*$

D. Deriving the Differential Equations of Motion

1) Case of $q_1=z_1^*(t)$ *:*

$$\frac{d}{dt}\left[\frac{\partial L}{\partial\dot{z}_1^*(t)}\right]=m_1.\ddot{z}_1^*(t),\ \frac{\partial L}{\partial z_1^*(t)}=-(k_{EV}+2k_V)z_1^*(t)+2k_V.z_2^*(t)+k_{EV}.U_z(t)$$

$$\frac{\partial\Re}{\partial\dot{z}_1^*(t)}=(r_{EV}+2r_V).\dot{z}_1^*(t)-2r_V.\dot{z}_2^*(t)-r_{EV}.\dot{U}_z(t),\ \text{and}\ Q_{z_1^*}=0$$

Substitute from the equations of case (a) in Eq. 13, the first differential equation of motion can be obtained

$$m_1.\ddot{z}_1^*(t)+(r_{EV}+2r_V).\dot{z}_1^*(t)-2r_V.\dot{z}_2^*(t)+(k_{EV}+2k_V).z_1^*(t)-2k_Vz_2^*(t)=k_{EV}.U_z(t)+r_{EV}\dot{U}_z(t) \quad (18)$$

2) Case of $q_2=y_1^*(t)$ *:* $\frac{d}{dt}\left[\frac{\partial L}{\partial\dot{y}_1^*(t)}\right]=m_1.\ddot{y}_1^*(t),\ \frac{\partial L}{\partial y_1^*(t)}=-(2k_H+k_{EH}).y_1^*(t)+2k_H.y_2^*(t)-ck_H.\varphi_2^*(t)+k_{EH}U_y(t)$

$$\frac{\partial\Re}{\partial\dot{y}_1^*(t)}=(2r_H+r_{EH}).\dot{y}_1^*(t)-2r_H.\dot{y}_2^*(t)+cr_H.\dot{\varphi}_2^*(t)-r_{EH}\dot{U}_y(t),\ \text{and}\ Q_{y_1^*}=0$$

Substitute from the equations of case (b) in Eq. 13, the second differential equation of motion can be obtained

$$m_1.\ddot{y}_1^*(t)+(r_{EH}+2r_H).\dot{y}_1^*(t)-2r_H.\dot{y}_2^*(t)+cr_H.\dot{\varphi}_2^*(t)+(k_{EH}+2k_H).y_1^*(t)-2k_H.y_2^*(t)+ck_H.\varphi_2^*(t)=$$
$$k_{EH}U_y(t)+r_{EH}.\dot{U}_y(t) \quad (19)$$

3) Case of $q_3=\phi_1^*(t)$ *:* $\frac{d}{dt}\left[\frac{\partial L}{\partial\dot{\varphi}_1^*(t)}\right]=J_1.\ddot{\varphi}_1^*(t),\ \frac{\partial L}{\partial\varphi_1^*(t)}=-(k_{EK}+\frac{b^2}{2}k_V).\varphi_1^*(t)+\frac{b^2}{2}k_V.\varphi_2^*(t)$

$$\frac{\partial\Re}{\partial\dot{\varphi}_1^*(t)}=(r_{EK}+\frac{b^2}{2}r_V).\dot{\varphi}_1^*(t)-\frac{b^2}{2}r_V.\dot{\varphi}_2^*(t),\ \text{and}\ Q_{\dot{\varphi}_1^*}=0$$

Substitute from the equations of case (c) in Eq. 13, the third differential equation of motion can be obtained

$$J_1.\ddot{\varphi}_1^*(t)+[r_{EK}+0.5b^2.r_V].\dot{\varphi}_1^*(t)-0.5b^2.r_V.\dot{\varphi}_2^*(t)+(k_{EK}+0.5b^2.k_V).\varphi_1^*(t)-0.5b^2.k_V.\varphi_2^*(t)=0 \quad (20)$$

4) Case of $q_4=z_2^*(t)$ *:*

$$\frac{d}{dt}\left[\frac{\partial L}{\partial\dot{z}_2^*(t)}\right]=m_2.\ddot{z}_2^*(t),\ \frac{\partial L}{\partial z_2^*(t)}=2k_V.z_1^*(t)-2k_V.z_2^*(t)$$

$$\frac{\partial \Re}{\partial \dot{z}_2^*} = -2r_V.\dot{z}_1^*(t) + 2r_V.\dot{z}_2^*(t), \text{ and } Q_{z_2^*} = 0$$

Similarly, the fourth differential equation of motion can be obtained

$$m_2.\ddot{z}_2^*(t) - 2r_V.\dot{z}_1^*(t) + 2r_V.\dot{z}_2^*(t) - 2k_V.z_1^*(t) + 2k_V.z_2^*(t) = 0 \tag{21}$$

5) Case of $q_5 = y_2^*(t)$ *:*

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{y}_2^*(t)}\right] = m_2.\ddot{y}_2^*(t), \quad \frac{\partial L}{\partial y_2^*(t)} = 2k_H.y_1^*(t) - 2k_H.y_2^*(t) + ck_H.\varphi_2^*(t)$$

$$\frac{\partial \Re}{\partial \dot{y}_2^*(t)} = -2r_H.\dot{y}_1^*(t) + 2r_H.\dot{y}_2^*(t) - cr_H.\dot{\varphi}_2^*(t), \text{ and } Q_{y_2^*} = W(t)$$

Similarly, the fifth differential equation of motion can be obtained

$$m_2.\ddot{y}_2^*(t) - 2r_H.\dot{y}_1^*(t) + 2r_H.\dot{y}_2^*(t) - cr_H.\dot{\varphi}_2^* - 2k_H.y_1^*(t) + 2k_H.y_2^*(t) - ck_H.\varphi_2^* = W(t) \tag{22}$$

6) Case of $q_6 = \phi_2^*$ *:*

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\varphi}_2^*(t)}\right] = J_2.\ddot{\varphi}_2^*(t), \quad \frac{\partial L}{\partial \varphi_2^*(t)} = -ck_H.y_1^*(t) + \frac{b^2}{2}k_V.\varphi_1^*(t) + ck_H.y_2^*(t) - \left(\frac{c^2}{2}k_H + \frac{b^2}{2}k_V\right).\varphi_2^*(t)$$

$$\frac{\partial \Re}{\partial \dot{\varphi}_1^*(t)} = cr_H.\dot{y}_1^*(t) - \frac{b^2}{2}r_V.\dot{\varphi}_1^*(t) - cr_H.\dot{y}_2^*(t) + \left(\frac{b^2}{2}r_V + \frac{c^2}{2}r_H\right).\dot{\varphi}_2^*(t), \text{ and } Q_{\varphi_2^*} = M_W(t)$$

Similarly, the sixth differential equation of motion can be obtained

$$\begin{aligned} &J_2.\ddot{\varphi}_2^*(t) + cr_H.\dot{y}_1^*(t) - 0.5b^2.r_V.\dot{\varphi}_1^*(t) - cr_H.\dot{y}_2^*(t) + (0.5c^2.r_H + 0.5b^2.r_V).\dot{\varphi}_2^*(t) \\ &+ ck_H.y_1^*(t) - 0.5b^2.k_V.\varphi_1^*(t) - ck_H.y_2^*(t) + (0.5c^2.k_H + 0.5b^2.k_V).\varphi_2^*(t) = M_W(t) \end{aligned} \tag{23}$$

Equations 18-23 can be written in the following matrix form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{z}_1^* \\ \ddot{y}_1^* \\ \ddot{\varphi}_1^* \\ \ddot{z}_2^* \\ \ddot{y}_2^* \\ \ddot{\phi}_2^* \end{bmatrix} + \begin{bmatrix} \frac{(r_{EV} + 2r_V)}{m_1} & 0 & 0 & -\frac{2r_V}{m_1} & 0 & 0 \\ 0 & \frac{r_{EH} + 2r_H}{m_1} & 0 & 0 & -\frac{2r_H}{m_1} & \frac{cr_H}{m_1} \\ 0 & 0 & \frac{2r_{EK} + b^2 r_V}{2J_1} & 0 & 0 & -\frac{b^2 r_V}{2J_1} \\ -\frac{2r_V}{m_2} & 0 & 0 & \frac{2r_V}{m_2} & 0 & 0 \\ 0 & -\frac{2r_H}{m_2} & 0 & 0 & \frac{2r_H}{m_2} & -\frac{cr_H}{m_2} \\ 0 & \frac{cr_H}{J_2} & -\frac{b^2 r_V}{2J_2} & 0 & -\frac{cr_H}{J_2} & \frac{(c^2 r_H + b^2 r_V)}{2J_2} \end{bmatrix} \begin{bmatrix} \dot{z}_1^* \\ \dot{y}_1^* \\ \dot{\varphi}_1^* \\ \dot{z}_2^* \\ \dot{y}_2^* \\ \dot{\varphi}_2^* \end{bmatrix} +$$

$$\begin{bmatrix} \frac{k_{EV} + 2k_V}{m_1} & 0 & 0 & -\frac{2k_V}{m_1} & 0 & 0 \\ 0 & \frac{k_{EH} + 2k_H}{m_1} & 0 & 0 & -\frac{2k_H}{m_1} & \frac{ck_H}{m_1} \\ 0 & 0 & \frac{2k_{EK} + b^2 k_V}{2J_1} & 0 & 0 & -\frac{b^2 k_V}{2J_1} \\ -\frac{2k_V}{m_2} & 0 & 0 & \frac{2k_V}{m_2} & 0 & 0 \\ 0 & -\frac{2k_H}{m_2} & 0 & 0 & \frac{2k_H}{m_2} & -\frac{ck_H}{m_2} \\ 0 & \frac{ck_H}{J_2} & -\frac{b^2 k_V}{2J_2} & 0 & -\frac{ck_H}{J_2} & \frac{c^2 k_H + b^2 k_V}{2J_2} \end{bmatrix} \begin{bmatrix} z_1^* \\ y_1^* \\ \varphi_1^* \\ z_2^* \\ y_2^* \\ \varphi_2^* \end{bmatrix} =$$

$$\begin{bmatrix} \frac{k_{EV}}{m_1} & \frac{r_{EV}}{m_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{k_{EH}}{m_1} & \frac{r_{EH}}{m_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{m_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{J_2} \end{bmatrix} \begin{bmatrix} \xi(t) \\ \dot{\xi}(t) \\ \eta(t) \\ \dot{\eta}(t) \\ W(t) \\ M_W(t) \end{bmatrix} \tag{24}$$

E. Normalization of the System Differential Equations of Motion

The system differential equations of motion of the high tower building with its foundation can be presented in a dimensionless form using the following quantities

$$z_1(t) = \frac{z_1^*(t)}{z_o}, y_1(t) = \frac{y_1^*(t)}{y_o}, \varphi_1(t) = \frac{\varphi_1^*(t)}{\varphi_o},$$
$$z_2(t) = \frac{z_2^*(t)}{z_o}, y_2(t) = \frac{y_2^*(t)}{y_o}, \varphi_2(t) = \frac{\varphi_2^*(t)}{\varphi_o}, \xi(t) = \frac{\xi^*(t)}{\xi_o}, \eta(t) = \frac{\eta^*}{\eta_o}$$

Where $z_o = y_o = 1\ cm, \xi_o = \eta_o = 1\,cm\ \ and\ \ \varphi_o = 1\,rad$.

Applying the time normalization through the following transformations $\tau = \omega_o t$, $d\tau = \omega_o dt$, where $\omega_o = 1\ rad/s$ and $\frac{dz}{dt} = \omega_o \frac{dz}{d\tau}$, $\frac{d^2 z}{dt^2} = \omega_o^2 \frac{d^2 z}{d\tau^2}$, $\Omega_1 t = \frac{\Omega_1}{\omega_o}\tau = \eta_1 t$

Therefore the differential equations of motion will be written in the following dimensionless form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_1''(\tau) \\ y_1''(\tau) \\ \varphi_1''(\tau) \\ z_2''(\tau) \\ y_2''(\tau) \\ \varphi_2''(\tau) \end{bmatrix} + \begin{bmatrix} \frac{(r_{EV}+2r_V)}{m_1\omega_o} & 0 & 0 & -\frac{2r_V}{m_1\omega_o} & 0 & 0 \\ 0 & \frac{r_{EH}+2r_H}{m_1\omega_o} & 0 & 0 & -\frac{2r_H}{m_1\omega_o} & \frac{cr_H}{m_1\omega_o} \\ 0 & 0 & \frac{2r_{EK}+b^2 r_V}{2J_1\omega_o} & 0 & 0 & -\frac{b^2 r_V}{2J_1\omega_o} \\ -\frac{2r_V}{m_2\omega_o} & 0 & 0 & \frac{2r_V}{m_2\omega_o} & 0 & 0 \\ 0 & -\frac{2r_H}{m_2\omega_o} & 0 & 0 & \frac{2r_H}{m_2\omega_o} & -\frac{cr_H}{m_2\omega_o} \\ 0 & \frac{cr_H}{J_2\omega_o} & -\frac{b^2 r_V}{2J_2\omega_o} & 0 & -\frac{cr_H}{J_2\omega_o} & \frac{(c^2 r_H + b^2 r_V)}{2J_2\omega_o} \end{bmatrix} \begin{bmatrix} z_1'(\tau) \\ y_1'(\tau) \\ \varphi_1'(\tau) \\ z_2'(\tau) \\ y_2'(\tau) \\ \varphi_2'(\tau) \end{bmatrix} +$$

$$\begin{bmatrix} \frac{k_{EV}+2k_V}{m_1\omega_o^2} & 0 & 0 & -\frac{2k_V}{m_1\omega_o^2} & 0 & 0 \\ 0 & \frac{k_{EH}+2k_H}{m_1\omega_o^2} & 0 & 0 & -\frac{2k_H}{m_1\omega_o^2} & \frac{ck_H\varphi_o}{m_1\omega_o^2 y_o} \\ 0 & 0 & \frac{2k_{EK}+b^2 k_V}{2J_1\omega_o^2} & 0 & 0 & -\frac{b^2 k_V}{2J_1\omega_o^2} \\ -\frac{2k_V}{m_2\omega_o^2} & 0 & 0 & \frac{2k_V}{m_2\omega_o^2} & 0 & 0 \\ 0 & -\frac{2k_H}{m_2\omega_o^2} & 0 & 0 & \frac{2k_H}{m_2\omega_o^2} & -\frac{ck_H\varphi_o}{m_2\omega_o^2 y_o} \\ 0 & \frac{ck_H y_o}{J_2\omega_o^2\varphi_o} & -\frac{b^2 k_V}{2J_2\omega_o^2} & 0 & -\frac{ck_H y_o}{J_2\omega_o^2\varphi_o} & \frac{c^2 k_H + b^2 k_V}{2J_2\omega_o^2} \end{bmatrix} \begin{bmatrix} z_1(\tau) \\ y_1(\tau) \\ \varphi_1(\tau) \\ z_2(\tau) \\ y_2(\tau) \\ \varphi_2(\tau) \end{bmatrix} =$$

$$\begin{bmatrix} \frac{k_{EV}}{m_1\omega_o^2 z_o} & \frac{r_{EV}}{m_1\omega_o z_o} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{k_{EH}}{m_1\omega_o^2 y_o} & \frac{r_{EH}}{m_1\omega_o y_o} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{m_2\omega_o^2 y_o} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{J_2\omega_o^2\varphi_o} \end{bmatrix} \begin{bmatrix} \xi(\tau) \\ \dot{\xi}(\tau) \\ \eta(\tau) \\ \dot{\eta}(\tau) \\ W(\tau) \\ M_W(\tau) \end{bmatrix} \tag{25}$$

V. ANALYTICAL SOLUTIONS USING THE GENERAL MODAL ANALYSIS METHOD

A. Eigen Value Problem

Homogeneous differential equations without damping:

$$\underline{M}^*\underline{\ddot{x}}^*(t)+\underline{K}^*\underline{x}^*(t)=\underline{0} \tag{26}$$

Assume that the exponential solutions of Eqs. 26 have the form

$$\underline{x}^*(t)=\underline{\hat{x}}\,e^{i\omega t} \tag{27}$$

Applying the solutions of Eqs. 27 in Eqs. 26 leads to the general eigen value problem

$$(-\omega^2\underline{M}^*+\underline{K}^*)\,\underline{\hat{x}}\,e^{i\omega t}=\underline{0} \qquad \text{or} \qquad (\underline{A}-\omega^2\underline{I})\,\underline{\hat{x}}=\underline{0}$$

Where the matrix $\underline{A}$ has the form

$$\underline{A}=\underline{M}^{*-1}.\underline{K}^*=\begin{bmatrix} \frac{k_{EV}+2k_V}{m_1} & 0 & 0 & -\frac{2k_V}{m_1} & 0 & 0 \\ 0 & \frac{k_{EH}+2k_H}{m_1} & 0 & 0 & -\frac{2k_H}{m_1} & \frac{ck_H}{m_1} \\ 0 & 0 & \frac{2k_{EK}+b^2k_V}{2J_1} & 0 & 0 & -\frac{b^2k_V}{2J_1} \\ -\frac{2k_V}{m_2} & 0 & 0 & \frac{2k_V}{m_2} & 0 & 0 \\ 0 & -\frac{2k_H}{m_2} & 0 & 0 & \frac{2k_H}{m_2} & -\frac{ck_H}{m_2} \\ 0 & \frac{ck_H}{J_2} & -\frac{b^2k_V}{2J_2} & 0 & -\frac{ck_H}{J_2} & \frac{c^2k_H+b^2k_V}{2J_2} \end{bmatrix} \tag{28}$$

Using Equation 3 one can obtain 12 eigen values $(\pm\omega_1,\pm\omega_2,\pm\omega_3,\pm\omega_4,\pm\omega_5,\pm\omega_6)$ and 6 eigen vectors $(\underline{\hat{x}}_1,\underline{\hat{x}}_2,\underline{\hat{x}}_3,\underline{\hat{x}}_4,\underline{\hat{x}}_5,\underline{\hat{x}}_6)$.

B. Modal Matrix

The modal matrix has the form

$$\underline{\chi}=[\underline{\hat{x}}_1,\underline{\hat{x}}_2,\underline{\hat{x}}_3,\underline{\hat{x}}_4,\underline{\hat{x}}_5,\underline{\hat{x}}_6]=\begin{bmatrix} \hat{\chi}_{11} & \hat{\chi}_{12} & \hat{\chi}_{13} & \hat{\chi}_{14} & \hat{\chi}_{15} & \hat{\chi}_{16} \\ \hat{\chi}_{21} & \hat{\chi}_{22} & \hat{\chi}_{23} & \hat{\chi}_{24} & \hat{\chi}_{25} & \hat{\chi}_{26} \\ \hat{\chi}_{31} & \hat{\chi}_{32} & \hat{\chi}_{33} & \hat{\chi}_{34} & \hat{\chi}_{35} & \hat{\chi}_{36} \\ \hat{\chi}_{41} & \hat{\chi}_{42} & \hat{\chi}_{43} & \hat{\chi}_{44} & \hat{\chi}_{45} & \hat{\chi}_{46} \\ \hat{\chi}_{51} & \hat{\chi}_{52} & \hat{\chi}_{53} & \hat{\chi}_{54} & \hat{\chi}_{55} & \hat{\chi}_{56} \\ \hat{\chi}_{61} & \hat{\chi}_{62} & \hat{\chi}_{63} & \hat{\chi}_{64} & \hat{\chi}_{65} & \hat{\chi}_{66} \end{bmatrix},\ \underline{\chi}^T=\begin{bmatrix} \underline{\hat{x}}_1^T \\ \underline{\hat{x}}_2^T \\ \underline{\hat{x}}_3^T \\ \underline{\hat{x}}_4^T \\ \underline{\hat{x}}_5^T \\ \underline{\hat{x}}_6^T \end{bmatrix}=\begin{bmatrix} \hat{\chi}_{11} & \hat{\chi}_{21} & \hat{\chi}_{31} & \hat{\chi}_{41} & \hat{\chi}_{51} & \hat{\chi}_{61} \\ \hat{\chi}_{12} & \hat{\chi}_{22} & \hat{\chi}_{32} & \hat{\chi}_{42} & \hat{\chi}_{52} & \hat{\chi}_{62} \\ \hat{\chi}_{13} & \hat{\chi}_{23} & \hat{\chi}_{33} & \hat{\chi}_{43} & \hat{\chi}_{53} & \hat{\chi}_{63} \\ \hat{\chi}_{14} & \hat{\chi}_{24} & \hat{\chi}_{34} & \hat{\chi}_{44} & \hat{\chi}_{54} & \hat{\chi}_{64} \\ \hat{\chi}_{15} & \hat{\chi}_{25} & \hat{\chi}_{35} & \hat{\chi}_{45} & \hat{\chi}_{55} & \hat{\chi}_{65} \\ \hat{\chi}_{16} & \hat{\chi}_{26} & \hat{\chi}_{36} & \hat{\chi}_{46} & \hat{\chi}_{56} & \hat{\chi}_{66} \end{bmatrix}$$

C. Decoupling of the System Differential Equations

The transformation of coordinates can be carried out using the equation

$$\underline{x}^*=\underline{\chi}\cdot\underline{q}$$

and the system of the vibration differential equations will has the form

$$\underline{\chi}^T\,\underline{M}^*\,\underline{\chi}\,\underline{\ddot{q}}+\underline{\chi}^T\,\underline{R}^*\,\underline{\chi}\,\underline{\dot{q}}+\underline{\chi}^T\,\underline{K}^*\,\underline{\chi}\,\underline{q}=\underline{\chi}^T\,\underline{B}^*\underline{U}^*$$

Where $\underline{\chi}^T\,\underline{M}^*\,\underline{\chi}=\underline{I}$, $\underline{\chi}^T\,\underline{R}^*\,\underline{\chi}\approx diag.[2\,D\omega]$, $\underline{\chi}^T\,\underline{K}^*\,\underline{\chi}\approx diag.[\,\omega^2\,]$, and $\underline{\chi}^T\,\underline{F}^*(t)\approx diag.[\,\omega^2\,]\,\underline{Q}$

When the damping forces of the equivalent system are smaller than its elastic restoring forces, then the coupled terms of the transformed damping matrix can be neglected without any great error. The decoupled differential equations of the system will have the form

$$\underline{I}\,\underline{\ddot{q}}+diag.[2D\omega]\,\underline{\dot{q}}+diag.[\,\omega^2\,]\,\underline{q}=diag.[\,\omega^2\,]\,\underline{Q}$$

$$\ddot{q}_n(t)+2D_n\omega_n\,\dot{q}_n(t)+\omega_n^2\,q_n(t)=\omega_n^2\,Q_n(t)\ ,\qquad n = 1, 2, \ldots, 6$$

$$\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}\ddot{q}_1(t)\\\ddot{q}_2(t)\\\ddot{q}_3(t)\\\ddot{q}_4(t)\\\ddot{q}_5(t)\\\ddot{q}_6(t)\end{bmatrix}+\begin{bmatrix}2D_1\omega_1&0&0&0&0&0\\0&2D_2\omega_2&0&0&0&0\\0&0&2D_3\omega_3&0&0&0\\0&0&0&2D_4\omega_4&0&0\\0&0&0&0&2D_5\omega_5&0\\0&0&0&0&0&2D_6\omega_6\end{bmatrix}\begin{bmatrix}\dot{q}_1(t)\\\dot{q}_2(t)\\\dot{q}_3(t)\\\dot{q}_4(t)\\\dot{q}_5(t)\\\dot{q}_6(t)\end{bmatrix}+$$

$$\begin{bmatrix}\omega_1^2&0&0&0&0&0\\0&\omega_2^2&0&0&0&0\\0&0&\omega_3^2&0&0&0\\0&0&0&\omega_4^2&0&0\\0&0&0&0&\omega_5^2&0\\0&0&0&0&0&\omega_6^2\end{bmatrix}\begin{bmatrix}q_1(t)\\q_2(t)\\q_3(t)\\q_4(t)\\q_5(t)\\q_6(t)\end{bmatrix}=\begin{bmatrix}\omega_1^2&0&0&0&0&0\\0&\omega_2^2&0&0&0&0\\0&0&\omega_3^2&0&0&0\\0&0&0&\omega_4^2&0&0\\0&0&0&0&\omega_5^2&0\\0&0&0&0&0&\omega_6^2\end{bmatrix}\begin{bmatrix}Q_1(t)\\Q_2(t)\\Q_3(t)\\Q_4(t)\\Q_5(t)\\Q_6(t)\end{bmatrix}\tag{29}$$

The general external excitations of the system are

$$\underline{Q}(t)=diag.[\,\omega^2\,]^{-1}.\,\underline{\chi}^T\,\underline{B}^*\underline{U}^*(\,t\,)$$

$$=\begin{bmatrix}1/\omega_1^2&0&0&0&0&0\\0&1/\omega_2^2&0&0&0&0\\0&0&1/\omega_3^2&0&0&0\\0&0&0&1/\omega_4^2&0&0\\0&0&0&0&1/\omega_5^2&0\\0&0&0&0&0&1/\omega_6^2\end{bmatrix}\begin{bmatrix}\hat{\chi}_{11}&\hat{\chi}_{21}&\hat{\chi}_{31}&\hat{\chi}_{41}&\hat{\chi}_{51}&\hat{\chi}_{61}\\\hat{\chi}_{12}&\hat{\chi}_{22}&\hat{\chi}_{32}&\hat{\chi}_{42}&\hat{\chi}_{52}&\hat{\chi}_{62}\\\hat{\chi}_{13}&\hat{\chi}_{23}&\hat{\chi}_{33}&\hat{\chi}_{43}&\hat{\chi}_{53}&\hat{\chi}_{63}\\\hat{\chi}_{14}&\hat{\chi}_{24}&\hat{\chi}_{34}&\hat{\chi}_{44}&\hat{\chi}_{54}&\hat{\chi}_{64}\\\hat{\chi}_{15}&\hat{\chi}_{25}&\hat{\chi}_{35}&\hat{\chi}_{45}&\hat{\chi}_{55}&\hat{\chi}_{65}\\\hat{\chi}_{16}&\hat{\chi}_{26}&\hat{\chi}_{36}&\hat{\chi}_{46}&\hat{\chi}_{56}&\hat{\chi}_{66}\end{bmatrix}\begin{bmatrix}k_{EV}&r_{EV}&0&0&0&0\\0&0&k_{EH}&r_{EH}&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}\xi(t)\\\dot{\xi}(t)\\\eta(t)\\\dot{\eta}(t)\\W(t)\\M_W(t)\end{bmatrix}$$

$$\underline{Q}(t)=diag.[\frac{1}{\omega^2}]\,.\,\underline{\chi}^T.\,\underline{F}^*(\,t\,)\quad\text{and}\quad Q_n(t)=\frac{1}{\omega_n^2}.\,\chi_n^T.\,F_n(\,t\,)\quad,\quad n = 1, 2, \ldots, 6$$

$$=\begin{bmatrix}(1/\omega_1^2)\hat{\chi}_{11}&(1/\omega_1^2)\hat{\chi}_{21}&(1/\omega_1^2)\hat{\chi}_{31}&(1/\omega_1^2)\hat{\chi}_{41}&(1/\omega_1^2)\hat{\chi}_{51}&(1/\omega_1^2)\hat{\chi}_{61}\\(1/\omega_2^2)\hat{\chi}_{12}&(1/\omega_2^2)\hat{\chi}_{22}&(1/\omega_2^2)\hat{\chi}_{32}&(1/\omega_2^2)\hat{\chi}_{42}&(1/\omega_2^2)\hat{\chi}_{52}&(1/\omega_2^2)\hat{\chi}_{62}\\(1/\omega_3^2)\hat{\chi}_{13}&(1/\omega_3^2)\hat{\chi}_{23}&(1/\omega_3^2)\hat{\chi}_{33}&(1/\omega_3^2)\hat{\chi}_{43}&(1/\omega_3^2)\hat{\chi}_{53}&(1/\omega_3^2)\hat{\chi}_{63}\\(1/\omega_4^2)\hat{\chi}_{14}&(1/\omega_4^2)\hat{\chi}_{24}&(1/\omega_4^2)\hat{\chi}_{34}&(1/\omega_4^2)\hat{\chi}_{44}&(1/\omega_4^2)\hat{\chi}_{54}&(1/\omega_4^2)\hat{\chi}_{64}\\(1/\omega_5^2)\hat{\chi}_{15}&(1/\omega_5^2)\hat{\chi}_{25}&(1/\omega_5^2)\hat{\chi}_{35}&(1/\omega_5^2)\hat{\chi}_{45}&(1/\omega_5^2)\hat{\chi}_{55}&(1/\omega_5^2)\hat{\chi}_{65}\\(1/\omega_6^2)\hat{\chi}_{16}&(1/\omega_6^2)\hat{\chi}_{26}&(1/\omega_6^2)\hat{\chi}_{36}&(1/\omega_6^2)\hat{\chi}_{46}&(1/\omega_6^2)\hat{\chi}_{56}&(1/\omega_6^2)\hat{\chi}_{66}\end{bmatrix}\begin{bmatrix}k_{EV}&r_{EV}&0&0&0&0\\0&0&k_{EH}&r_{EH}&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}\xi(t)\\\dot{\xi}(t)\\\eta(t)\\\dot{\eta}(t)\\W(t)\\M_W(t)\end{bmatrix}$$

$$Q_n(t)=\begin{bmatrix}(1/\omega_1^2)\hat{\chi}_{11}.k_{EV}&(1/\omega_1^2)\hat{\chi}_{11}.r_{EV}&(1/\omega_1^2)\hat{\chi}_{21}.k_{EH}&(1/\omega_1^2)\hat{\chi}_{21}.r_{EH}&(1/\omega_1^2)\hat{\chi}_{51}.1&(1/\omega_1^2)\hat{\chi}_{61}.1\\(1/\omega_2^2)\hat{\chi}_{12}.k_{EV}&(1/\omega_2^2)\hat{\chi}_{12}.r_{EV}&(1/\omega_2^2)\hat{\chi}_{22}.k_{EH}&(1/\omega_2^2)\hat{\chi}_{22}.r_{EH}&(1/\omega_2^2)\hat{\chi}_{52}.1&(1/\omega_2^2)\hat{\chi}_{62}.1\\(1/\omega_3^2)\hat{\chi}_{13}.k_{EV}&(1/\omega_3^2)\hat{\chi}_{13}.r_{EV}&(1/\omega_3^2)\hat{\chi}_{23}.k_{EH}&(1/\omega_3^2)\hat{\chi}_{23}.r_{EH}&(1/\omega_3^2)\hat{\chi}_{53}.1&(1/\omega_3^2)\hat{\chi}_{63}.1\\(1/\omega_4^2)\hat{\chi}_{14}.k_{EV}&(1/\omega_4^2)\hat{\chi}_{14}.r_{EV}&(1/\omega_4^2)\hat{\chi}_{24}.k_{EH}&(1/\omega_4^2)\hat{\chi}_{24}.r_{EH}&(1/\omega_4^2)\hat{\chi}_{54}.1&(1/\omega_4^2)\hat{\chi}_{64}.1\\(1/\omega_5^2)\hat{\chi}_{15}.k_{EV}&(1/\omega_5^2)\hat{\chi}_{15}.r_{EV}&(1/\omega_5^2)\hat{\chi}_{25}.k_{EH}&(1/\omega_5^2)\hat{\chi}_{25}.r_{EH}&(1/\omega_5^2)\hat{\chi}_{55}.1&(1/\omega_5^2)\hat{\chi}_{65}.1\\(1/\omega_6^2)\hat{\chi}_{16}.k_{EV}&(1/\omega_6^2)\hat{\chi}_{16}.r_{EV}&(1/\omega_6^2)\hat{\chi}_{26}.k_{EH}&(1/\omega_6^2)\hat{\chi}_{26}.r_{EH}&(1/\omega_6^2)\hat{\chi}_{56}.1&(1/\omega_6^2)\hat{\chi}_{66}.1\end{bmatrix}\begin{bmatrix}\xi(t)\\\dot{\xi}(t)\\\eta(t)\\\dot{\eta}(t)\\W(t)\\M_W(t)\end{bmatrix}\tag{30}$$

$$Q_n(t)=\frac{1}{\omega_n^2}\,[B_{n1}\xi(\,t\,)+B_{n2}\dot{\xi}(\,t\,)+B_{n3}\eta(\,t\,)+B_{n4}\dot{\eta}(\,t\,)+B_{n5}W(\,t\,)+B_{n6}M_W(\,t\,)]$$

Applying the total turbulent wind forces W(t) in y-direction and the total wind moments $M_W(t)$ on the previous equations.

$$Q_n(t)=\frac{1}{\omega_n^2}[B_{n1}\xi(t)+B_{n2}\dot{\xi}(t)+B_{n3}\eta(t)+B_{n4}\dot{\eta}(t)+B_{n5}\int^A C_p.\rho.\overline{U}(z).U'(z,t)\,dA+$$
$$B_{n6}\int^A (z-\frac{c}{2}).C_p.\rho.\overline{U}(z).U'(z,t)\,dA] \tag{31}$$

The decoupled system of differential equations can be presented in the following form

$$\left.\begin{aligned}
m_1\ddot{q}_1(t)+r_1\dot{q}_1(t)+k_1q_1(t)&=\chi_{11}f_1(t)+\chi_{21}f_2(t)+\chi_{31}f_3(t)+\chi_{41}f_4(t)+\chi_{51}f_5(t)+\chi_{61}f_6(t)\\
m_2\ddot{q}_2(t)+r_2\dot{q}_2(t)+k_2q_2(t)&=\chi_{12}f_1(t)+\chi_{22}f_2(t)+\chi_{32}f_3(t)+\chi_{42}f_4(t)+\chi_{52}f_5(t)+\chi_{62}f_6(t)\\
m_3\ddot{q}_3(t)+r_3\dot{q}_3(t)+k_3q_3(t)&=\chi_{13}f_1(t)+\chi_{23}f_2(t)+\chi_{33}f_3(t)+\chi_{43}f_4(t)+\chi_{53}f_5(t)+\chi_{63}f_6(t)\\
m_4\ddot{q}_4(t)+r_4\dot{q}_4(t)+k_4q_4(t)&=\chi_{14}f_1(t)+\chi_{24}f_2(t)+\chi_{34}f_3(t)+\chi_{44}f_4(t)+\chi_{54}f_5(t)+\chi_{64}f_6(t)\\
m_5\ddot{q}_5(t)+r_5\dot{q}_5(t)+k_5q_5(t)&=\chi_{15}f_1(t)+\chi_{25}f_2(t)+\chi_{35}f_3(t)+\chi_{45}f_4(t)+\chi_{55}f_5(t)+\chi_{65}f_6(t)\\
m_6\ddot{q}_6(t)+r_6\dot{q}_6(t)+k_6q_6(t)&=\chi_{16}f_1(t)+\chi_{26}f_2(t)+\chi_{36}f_3(t)+\chi_{46}f_4(t)+\chi_{56}f_5(t)+\chi_{66}f_6(t)
\end{aligned}\right\} \tag{32}$$

$$\ddot{q}_i(t)+(\frac{r_i}{m_i})\dot{q}_i(t)+(\frac{k_i}{m_i})q_i(t)=(\frac{\chi_{1i}}{m_i})[k_{EV}\xi(t)+r_{EV}\dot{\xi}(t)]+(\frac{\chi_{2i}}{m_i})[k_{EH}\eta(t)+r_{EH}\dot{\eta}(t)]+(\frac{\chi_{3i}}{m_i})f_3(t)+$$
$$(\frac{\chi_{4i}}{m_i})f_4(t)+(\frac{\chi_{5i}}{m_i})f_5(t)+(\frac{\chi_{6i}}{m_i})f_6(t) \tag{33}$$

From the previous equations, one can obtain the following imaginary transformation functions

$$H_1(\Omega)=\frac{\frac{\chi_{1i}}{m_i}[k_{EV}+j\,r_{EV}\Omega]}{(-\Omega^2+\omega_i^2)+j(2D_i\omega_i\Omega)}=\frac{\frac{\chi_{1i}}{m_i}.\frac{1}{\omega_i^2}[k_{EV}+j\,r_{EV}\Omega]}{[1-(\frac{\Omega}{\omega_i})^2]+j(2D_i\frac{\Omega}{\omega_i})}=\frac{\frac{\chi_{1i}}{k_i}[k_{EV}+j\,r_{EV}\Omega]}{[1-(\frac{\Omega}{\omega_i})^2]+j(2D_i\frac{\Omega}{\omega_i})}$$

$$\left.H_2(\Omega)=\frac{\frac{\chi_{2i}}{k_i}[k_{EH}+j\,r_{EH}\Omega]}{[1-(\frac{\Omega}{\omega_i})^2]+j(2D_i\frac{\Omega}{\omega_i})},\ H_3(\Omega)=\frac{\frac{\chi_{3i}}{k_i}.1}{[1-(\frac{\Omega}{\omega_i})^2]+j(2D_i\frac{\Omega}{\omega_i})},\ H_4(\Omega)=\frac{\frac{\chi_{4i}}{k_i}.1}{[1-(\frac{\Omega}{\omega_i})^2]+j(2D_i\frac{\Omega}{\omega_i})}\right\} \tag{34}$$

$$H_5(\Omega)=\frac{\frac{\chi_{5i}}{k_i}.1}{[1-(\frac{\Omega}{\omega_i})^2]+j(2D_i\frac{\Omega}{\omega_i})},\ H_6(\Omega)=\frac{\frac{\chi_{6i}}{k_i}.1}{[1-(\frac{\Omega}{\omega_i})^2]+j(2D_i\frac{\Omega}{\omega_i})},\quad \text{where } \frac{r_i}{m_i}=2D_i\omega_i \text{ and } \frac{k_i}{m_i}=\omega_i^2$$

A dynamical system with known properties responds to a dynamical loading in a known manner, providing the time-description of the loading is available a priori. Such description is however not possible in case of the excitations due to earthquake ground motions or fluctuating wind loads. Therefore, the safety of a structural system has to be ensured by stochastic modeling of these motions for perceived seismic hazard at the site of the system and by predicting the structural response in probabilistic sense with the help of well-known concepts of random vibration theory. This theory estimates the statistical variations in the peak structural response due to possible variations in the time-description of the excitation (there may be several `different looking' time-histories corresponding to a given characterization of the excitation). The classical random vibration theory makes use of the frequency distribution of input energy as obtained from the Fourier Transform of the excitation. However, since Fourier Transform gives only an `average' energy distribution in an excitation with time-evolving structure, this theory is insufficient for those cases where the non-stationary processes cannot be modeled as stationary or quasi-stationary. As a natural extension to double Fourier Transform for such processes is not considered to be practical, a large amount of effort has been devoted to modeling a (slowly-varying) non-stationary process through modulating function-based power spectral density function (PSDF). The auto power spectral density function of the response as a result of random wind and earthquake excitations with respect to general coordinates has the form.

$$S_{q_iq_i}(\Omega)=\sum_{r=1}^{6}\sum_{s=1}^{6}H_r^*(\Omega)H_s(\Omega)S_{f_rf_s}(\Omega)$$
$$S_{q_iq_i}(\Omega)=H_1^*(\Omega)H_1(\Omega)S_{\xi\xi}(\Omega)+H_1^*(\Omega)H_2(\Omega)S_{\xi\eta}(\Omega)+......+H_1^*(\Omega)H_6(\Omega)S_{\xi f_6}(\Omega)+$$

$$H_2^*(\Omega)H_1(\Omega)S_{\eta\xi}(\Omega)+H_2^*(\Omega)H_2(\Omega)S_{\eta\eta}(\Omega)+......+H_2^*(\Omega)H_6(\Omega)S_{\eta f_6}(\Omega)+......+$$
$$H_6^*(\Omega)H_1(\Omega)S_{f_6\xi}(\Omega)+H_6^*(\Omega)H_2(\Omega)S_{f_6\eta}(\Omega)+......+H_6^*(\Omega)H_6(\Omega)S_{f_6 f_6}(\Omega) \quad (35)$$

The cross correlation function of excitation functions with respect to general coordinates is

$$R_{Q_iQ_j}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} Q_i(t)\,Q_j(t+\tau)\,dt$$

$$=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}[\chi_{1i}f_1(t)+\chi_{2i}f_2(t)+...+\chi_{6i}f_6(t)][\chi_{1j}f_1(t+\tau)+\chi_{2j}f_2(t+\tau)+...+\chi_{6j}f_6(t+\tau)]\,dt$$
$$=\chi_{1i}\chi_{1j}R_{f_1f_1}(\tau)+\chi_{1i}\chi_{2j}R_{f_1f_2}(\tau)+...+\chi_{1i}\chi_{6j}R_{f_1f_6}(\tau)+\chi_{2i}\chi_{1j}R_{f_2f_1}(\tau)+\chi_{2i}\chi_{2j}R_{f_2f_2}(\tau)$$
$$+...+\chi_{2i}\chi_{6j}R_{f_2f_6}(\tau)+...+\chi_{6i}\chi_{1j}R_{f_6f_1}(\tau)+\chi_{6i}\chi_{2j}R_{f_6f_2}(\tau)+...+\chi_{6i}\chi_{6j}R_{f_6f_6}(\tau) \quad (36)$$

The cross and auto power spectral density functions of excitation functions are

$$S_{Q_iQ_j}(\Omega)=\sum_{k=1}^{N}\sum_{l=1}^{N}\chi_{ki}\,\chi_{lj}\,S_{f_kf_l}(\Omega),\; S_{f_kf_l}(\Omega)=HA_k^*(\Omega).HA_l(\Omega).S_{kl}(\Omega),\; S_{f_1f_1}(\Omega)=HA_1^*(\Omega).HA_1(\Omega).S_{\xi\xi}(\Omega) \quad (37)$$

$$f_1(t)=u(1).\xi(t)+u(2).\dot{\xi}(t) \quad ,$$
$$f_1(\Omega)=u(1).\xi(\Omega)+i\Omega\,u(2).\xi(\Omega)=[u(1)+i\Omega\,u(2)].\xi(\Omega)=[u(1)+i\Omega\,u(2)].[\dot{\xi}(\Omega)/i\Omega]$$

The excitation functions can be represented as

$$Q_i(t)=\sum_{n=1}^{6}\chi_{ni}\,f_n(t),\; Q_r(t)=\sum_{i=1}^{6}\chi_{ir}\,f_i(t),\; Q_s(t)=\sum_{j=1}^{6}\chi_{js}\,f_j(t),\; Q_r(t)\,Q_s(t+\tau)=\sum_{i=1}^{6}\sum_{j=1}^{6}\chi_{ir}\,f_i(t).\chi_{js}\,f_j(t) \quad (38)$$

$$Q_1(t)=\frac{1}{\omega_1^2}.\underline{\chi}_{(1)}^T\begin{bmatrix}f_1(t)\\f_2(t)\\f_3(t)\\f_4(t)\\f_5(t)\\f_6(t)\end{bmatrix}=\frac{1}{\omega_1^2}.\underline{\chi}_{(1)}^T\begin{bmatrix}k_{EV}\xi(t)+r_{EV}\dot{\xi}(t)\\k_{EH}\eta(t)+r_{EH}\dot{\eta}(t)\\0\\0\\\int^{A}C_p.\rho.\overline{U}(z).U'(z,t)\,dA\\\int^{A}(z-\frac{c}{2}).C_p.\rho.\overline{U}(z).U'(z,t)\,dA\end{bmatrix}=\frac{1}{\omega_1^2}.\underline{\chi}_{(1)}^T\begin{bmatrix}u(1)\xi(t)+u(2)\dot{\xi}(t)\\u(3)\eta(t)+u(4)\dot{\eta}(t)\\0\\0\\u(5)v(t)\\u(6)w(t)\end{bmatrix}$$

$$Q_2(t)=\frac{1}{\omega_2^2}.\underline{\chi}_{(2)}^T.\underline{f}(t),\; Q_3(t)=\frac{1}{\omega_3^2}.\underline{\chi}_{(3)}^T.\underline{f}(t),\; Q_4(t)=\frac{1}{\omega_4^2}.\underline{\chi}_{(4)}^T.\underline{f}(t),\; Q_5(t)=\frac{1}{\omega_5^2}.\underline{\chi}_{(5)}^T.\underline{f}(t),\; Q_6(t)=\frac{1}{\omega_6^2}.\underline{\chi}_{(6)}^T.\underline{f}(t) \quad (39)$$

The cross correlation function of excitations is

$$R_{Q_rQ_s}(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}Q_r(t)Q_s(t+\tau)\,dt=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}[\frac{1}{\omega_r^2}\underline{\chi}_{(r)}^T f(t)].[\frac{1}{\omega_s^2}\underline{\chi}_{(s)}^T f(t+\tau)]\,dt$$
$$=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{\omega_r^2}\frac{1}{\omega_s^2}\underline{\chi}_{ir}\underline{\chi}_{js}f_i(t)f_j(t+\tau)\,dt=\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{\omega_r^2}\frac{1}{\omega_s^2}\underline{\chi}_{ir}\underline{\chi}_{js}\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}f_i(t)f_j(t+\tau)\,dt$$
$$=\frac{1}{\omega_r^2}\frac{1}{\omega_s^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\underline{\chi}_{ir}\underline{\chi}_{js}\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}f_i(t)f_j(t+\tau)\,dt=\frac{1}{\omega_r^2}\frac{1}{\omega_s^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\underline{\chi}_{ir}\underline{\chi}_{js}.R_{f_if_j}(\tau)$$
$$=\frac{1}{\omega_r^2}\frac{1}{\omega_s^2}.[\chi_{1r}\chi_{1s}.R_{f_1f_1}(\tau)+\chi_{5r}\chi_{5s}.R_{f_5f_5}(\tau)+\chi_{5r}\chi_{6s}.R_{f_5f_6}(\tau)+\chi_{6r}\chi_{5s}.R_{f_6f_5}(\tau)+\chi_{6r}\chi_{6s}.R_{f_6f_6}(\tau)]$$
$$=\frac{1}{\omega_r^2}\frac{1}{\omega_s^2}.\{\chi_{1r}\chi_{1s}[k_{EV}^2.R_{\xi\xi}(\tau)+k_{EV}.r_{EV}.R_{\xi\dot{\xi}}(\tau)+r_{EV}.k_{EV}.R_{\dot{\xi}\xi}(\tau)+r_{EV}^2.R_{\dot{\xi}\dot{\xi}}(\tau)]+$$
$$[\chi_{5r}\chi_{5s}.R_{f_5f_5}(\tau)+\chi_{5r}\chi_{6s}.R_{f_5f_6}(\tau)+\chi_{6r}\chi_{5s}.R_{f_6f_5}(\tau)+\chi_{6r}\chi_{6s}.R_{f_6f_6}(\tau)]\}$$

The cross power spectral density function of excitation functions has the form

$$S_{Q_rQ_s}(\Omega)=\frac{1}{\omega_r^2}\frac{1}{\omega_s^2}.\{\chi_{1r}\chi_{1s}[k_{EV}^2.S_{\xi\xi}(\tau)+k_{EV}.r_{EV}.S_{\xi\dot{\xi}}(\tau)+r_{EV}.k_{EV}.S_{\dot{\xi}\xi}(\tau)+r_{EV}^2.S_{\dot{\xi}\dot{\xi}}(\tau)]+$$

$$C_f^2.\rho^2.\overline{U}^2(z).S_{uu}(\Omega)[\chi_{5r}\chi_{5s}.\left|X_{11}(\Omega)\right|^2+\chi_{5r}\chi_{6s}.\left|X_{12}(\Omega)\right|^2+\chi_{6r}\chi_{5s}.\left|X_{21}(\Omega)\right|^2+\chi_{6r}\chi_{6s}.\left|X_{22}(\Omega)\right|^2]\}$$

$$S_{Q_rQ_s}(\Omega)=\frac{1}{\omega_r^2}\frac{1}{\omega_s^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\underline{\chi}_{ir}\,\underline{\chi}_{js}.S_{f_if_j}(\Omega) \tag{40}$$

The differential equations of motion can be written in the form

$$\ddot{q}_i(t)+\frac{R_i}{m_i}\dot{q}_i(t)+\frac{k_i}{m_i}q_i(t)=\frac{1}{m_i}.Q_i^{'}(t)=\omega_i^2.Q_i(t)$$

$$\ddot{q}_i(t)+2D_i\omega_i.\dot{q}_i(t)+\omega_i^2.q_i(t)=\frac{1}{k_i}\omega_i^2.Q_i^{'}(t)=\omega_i^2\frac{1}{k_i}.Q_i^{'}(t)=\omega_i^2.Q_i(t)\text{ with }Q_i(t)=\frac{1}{\omega_i^2}\frac{1}{m_i}.Q_i^{'}(t) \tag{41}$$

The cross power spectral density function of the vibration response with respect to general coordinates is

$$S_{q_rq_s}(\Omega)=H_r^*(\Omega).H_s(\Omega).S_{Q_rQ_s}(\Omega) \tag{42}$$

The cross power spectral density function of the vibration response with respect to original coordinates is

$$S_{X_rX_s}(\Omega)=\sum_{i=1}^{n}\sum_{j=1}^{n}\chi_{ri}\chi_{sj}.S_{q_iq_j}(\Omega) \tag{43}$$

Substitute from Eq. 42 in Eq. 46 results in

$$S_{X_rX_s}(\Omega)=\sum_{i=1}^{n}\sum_{j=1}^{n}\chi_{ri}\chi_{sj}.H_i^*(\Omega).H_j(\Omega).S_{Q_iQ_j}(\Omega) \tag{44}$$

Substitute from Eq. 40 in Eq. 44, one can obtain the cross power spectral density function of the response with respect to original coordinates of the form

$$S_{X_rX_s}(\Omega)=\sum_{i=1}^{n}\sum_{j=1}^{n}\chi_{ri}\chi_{sj}.H_i^*(\Omega).H_j(\Omega).(\frac{1}{m_i}.\frac{1}{\omega_i^2}).(\frac{1}{m_j}.\frac{1}{\omega_j^2}).\sum_{k=1}^{n}\sum_{l=1}^{n}\chi_{ki}\chi_{lj}.S_{f_kf_l}(\Omega) \tag{45}$$

and the auto power spectral density function of the response with respect to original coordinates of the form

$$S_{X_nX_n}(\Omega)=\sum_{i=1}^{6}\sum_{j=1}^{6}\chi_{ni}\chi_{nj}.H_i^*(\Omega).H_j(\Omega).\frac{1}{k_i}.\frac{1}{k_j}\sum_{r=1}^{6}\sum_{s=1}^{6}\chi_{ri}\chi_{sj}.S_{f_rf_s}(\Omega) \tag{46}$$

D. The Power Spectral Density Function of the Excitations-Correlation Function of the Excitations

The correlation function of the excitations with respect to general coordinates is

$$R_{Q_rQ_s}(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}Q_r(t)\,Q_s(t+\tau)\,dt \tag{47}$$

Substitute from Eq. 32 in Eq. 47 results in

$$R_{Q_rQ_s}(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\frac{1}{\omega_r^2}[B_{r1}\xi(t)+B_{r2}\dot{\xi}(t)+B_{r3}\eta(t)+B_{r4}\dot{\eta}(t)+B_{r5}\int^{A}C_p.\rho.\overline{U}(z).U^{'}(z,t)\,dA+$$

$$B_{r6}\int^{A}(z-\frac{c}{2}).C_p.\rho.\overline{U}(z).U^{'}(z,t)\,dA].\frac{1}{\omega_s^2}[B_{s1}\xi(t+\tau)+B_{s2}\dot{\xi}(t+\tau)+B_{s3}\eta(t+\tau)+$$

$$B_{s4}\dot{\eta}(t+\tau)+B_{s5}\int^{A}C_p.\rho.\overline{U}(z).U^{'}(z,t+\tau)\,dA+B_{s6}\int^{A}(z-\frac{c}{2}).C_p.\rho.\overline{U}(z).U^{'}(z,t+\tau)\,dA]\,dt$$

$$R_{Q_rQ_s}(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\frac{1}{\omega_r^2}\frac{1}{\omega_s^2}\{\,B_{r1}B_{s1}\xi(t)\xi(t+\tau)+B_{r1}B_{s2}\xi(t)\dot{\xi}(t+\tau)+B_{r1}B_{s3}\xi(t)\eta(t+\tau)+B_{r1}B_{s4}\xi(t)\dot{\eta}(t+\tau)+$$

$$B_{r1}B_{s5}\xi(t)\int^{A}C_p.\rho.\overline{U}(z).U^{'}(z,t+\tau)\,dA+B_{r1}B_{s6}\xi(t)\int^{A}(z-\frac{c}{2}).C_p.\rho.\overline{U}(z).U^{'}(z,t+\tau)\,dA+$$

$$B_{r2}B_{s1}\dot{\xi}(t)\xi(t+\tau)+B_{r2}B_{s2}\dot{\xi}(t)\dot{\xi}(t+\tau)+B_{r2}B_{s3}\dot{\xi}(t)\eta(t+\tau)+B_{r2}B_{s4}\dot{\xi}(t)\dot{\eta}(t+\tau)+$$

$$B_{r2}B_{s5}\dot{\xi}(t)\int^{A} C_p.\rho.\overline{U}(z).U^{'}(z,t+\tau)\,dA+B_{r2}B_{s6}\dot{\xi}(t)\int^{A}(z-\frac{c}{2}).C_p.\rho.\overline{U}(z).U^{'}(z,t+\tau)\,dA+$$

$$B_{r3}B_{s1}\eta(t)\xi(t+\tau)+B_{r3}B_{s2}\eta(t)\dot{\xi}(t+\tau)+B_{r3}B_{s3}\eta(t)\eta(t+\tau)+B_{r3}B_{s4}\eta(t)\dot{\eta}(t+\tau)+$$

$$B_{r3}B_{s5}\eta(t)\int^{A} C_p.\rho.\overline{U}(z).U^{'}(z,t+\tau)\,dA+B_{r3}B_{s6}\eta(t)\int^{A}(z-\frac{c}{2}).C_p.\rho.\overline{U}(z).U^{'}(z,t+\tau)\,dA+$$

$$B_{r4}B_{s1}\dot{\eta}(t)\xi(t+\tau)+B_{r4}B_{s2}\dot{\eta}(t)\dot{\xi}(t+\tau)+B_{r4}B_{s3}\dot{\eta}(t)\eta(t+\tau)+B_{r4}B_{s4}\dot{\eta}(t)\dot{\eta}(t+\tau)+$$

$$B_{r4}B_{s5}\dot{\eta}(t)\int^{A} C_p.\rho.\overline{U}(z).U^{'}(z,t+\tau)\,dA+B_{r4}B_{s6}\dot{\eta}(t)\int^{A}(z-\frac{c}{2}).C_p.\rho.\overline{U}(z).U^{'}(z,t+\tau)\,dA+$$

$$B_{r5}B_{s1}[\int^{A} C_p.\rho.\overline{U}(z).U^{'}(z,t)\,dA].\xi(t+\tau)+B_{r5}B_{s2}[\int^{A} C_p.\rho.\overline{U}(z).U^{'}(z,t)\,dA].\dot{\xi}(t+\tau)+$$

$$B_{r5}B_{s3}[\int^{A} C_p.\rho.\overline{U}(z).U^{'}(z,t)\,dA].\eta(t+\tau)+B_{r5}B_{s4}[\int^{A} C_p.\rho.\overline{U}(z).U^{'}(z,t)\,dA].\dot{\eta}(t+\tau)+$$

$$B_{r5}B_{s5}[\int^{A} C_p.\rho.\overline{U}(z).U^{'}(z,t)\,dA].[\int^{A} C_p.\rho.\overline{U}(z).U^{'}(z,t+\tau)\,dA]+$$

$$B_{r5}B_{s6}[\int^{A} C_p.\rho.\overline{U}(z).U^{'}(z,t)\,dA].[\int^{A}(z-\frac{c}{2}).C_p.\rho.\overline{U}(z).U^{'}(z,t+\tau)\,dA]+$$

$$B_{r6}B_{s1}[\int^{A}(z-\frac{c}{2}).C_p.\rho.\overline{U}(z).U^{'}(z,t)\,dA].\xi(t+\tau)+B_{r6}B_{s2}[\int^{A}(z-\frac{c}{2}).C_p.\rho.\overline{U}(z).U^{'}(z,t)\,dA].\dot{\xi}(t+\tau)+$$

$$B_{r6}B_{s3}[\int^{A}(z-\frac{c}{2}).C_p.\rho.\overline{U}(z).U^{'}(z,t)\,dA].\eta(t+\tau)+B_{r6}B_{s4}[\int^{A}(z-\frac{c}{2}).C_p.\rho.\overline{U}(z).U^{'}(z,t)\,dA].\dot{\eta}(t+\tau)+$$

$$B_{r6}B_{s5}[\int^{A}(z-\frac{c}{2}).C_p.\rho.\overline{U}(z).U^{'}(z,t)\,dA].[\int^{A} C_p.\rho.\overline{U}(z).U^{'}(z,t+\tau)\,dA]+$$

$$B_{r6}B_{s6}[\int^{A}(z-\frac{c}{2}).C_p.\rho.\overline{U}(z).U^{'}(z,t)\,dA].[\int^{A}(z-\frac{c}{2}).C_p.\rho.\overline{U}(z).U^{'}(z,t+\tau)\,dA]\ \}dt$$

$$R_{Q_rQ_s}(\tau)=\frac{1}{\omega_r^2}\frac{1}{\omega_s^2}\ \{\ B_{r1}B_{s1}R_{\xi\xi}(\tau)+B_{r1}B_{s2}R_{\xi\dot{\xi}}(\tau)+B_{r1}B_{s3}R_{\xi\eta}(\tau)+B_{r1}B_{s4}R_{\xi\dot{\eta}}(\tau)+$$

$$B_{r1}B_{s5}\int^{A} C_p.\rho.\overline{U}(z).R_{\xi U^{'}}(\tau)\,dA+B_{r1}B_{s6}\int^{A}(z-\frac{c}{2}).C_p.\rho.\overline{U}(z).R_{\xi U^{'}}(\tau)\,dA+$$

$$B_{r2}B_{s1}R_{\dot{\xi}\xi}(\tau)+B_{r2}B_{s2}R_{\dot{\xi}\dot{\xi}}(\tau)+B_{r2}B_{s3}R_{\dot{\xi}\eta}(\tau)+B_{r2}B_{s4}R_{\dot{\xi}\dot{\eta}}(\tau)+$$

$$B_{r2}B_{s5}\int^{A} C_p.\rho.\overline{U}(z).R_{\dot{\xi}U^{'}}(\tau)\,dA+B_{r2}B_{s6}\int^{A}(z-\frac{c}{2}).C_p.\rho.\overline{U}(z).R_{\dot{\xi}U^{'}}(\tau)\,dA+$$

$$B_{r3}B_{s1}.R_{\eta\xi}(\tau)+B_{r3}B_{s2}.R_{\eta\dot{\xi}}(\tau)+B_{r3}B_{s3}R_{\eta\eta}(\tau)+B_{r3}B_{s4}.R_{\eta\dot{\eta}}(\tau)+$$

$$B_{r3}B_{s5}\int^{A} C_p.\rho.\overline{U}(z).\,R_{\eta U^{'}}(\tau)dA+B_{r3}B_{s6}\int^{A}(z-\frac{c}{2}).C_p.\rho.\overline{U}(z).R_{\eta U^{'}}(\tau)\,dA+$$

$$B_{r4}B_{s1}R_{\dot{\eta}\xi}(\tau)+B_{r4}B_{s2}R_{\dot{\eta}\dot{\xi}}(\tau)+B_{r4}B_{s3}R_{\dot{\eta}\eta}(\tau)+B_{r4}B_{s4}R_{\dot{\eta}\dot{\eta}}(\tau)+$$

$$B_{r4}B_{s5}\int^{A} C_p.\rho.\overline{U}(z).R_{\dot{\eta}U^{'}}(\tau)\,dA+B_{r4}B_{s6}\int^{A}(z-\frac{c}{2}).C_p.\rho.\overline{U}(z).R_{\dot{\eta}U^{'}}(\tau)\,dA+$$

$$B_{r5}B_{s1}\int^{A} C_p.\rho.\overline{U}(z).R_{U'\xi}(\tau)\,dA + B_{r5}B_{s2}\int^{A} C_p.\rho.\overline{U}(z).R_{U'\dot{\xi}}(\tau)\,dA + B_{r5}B_{s3}\int^{A} C_p.\rho.\overline{U}(z).R_{U'\eta}(\tau)\,dA +$$

$$B_{r5}B_{s4}\int^{A} C_p.\rho.\overline{U}(z).R_{U'\dot{\eta}}(\tau)\,dA + B_{r5}B_{s5}\int^{A}\int^{A} C_p^2.\rho^2.\overline{U}(z_1).\overline{U}(z_2).R_{U_1'U_2'}(\tau).dA_1.dA_2$$

$$B_{r5}B_{s6}\int^{A_1}\int^{A_2} C_p^2.\rho^2.\overline{U}(z_1).\overline{U}(z_2).(z_2-\frac{c}{2})R_{U_1'U_2'}(\tau).dA_1.dA_2 + B_{r6}B_{s1}\int^{A}(z-\frac{c}{2}).C_p.\rho.\overline{U}(z).R_{U'\xi}(\tau)\,dA +$$

$$B_{r6}B_{s2}\int^{A}(z-\frac{c}{2}).C_p.\rho.\overline{U}(z).R_{U'\dot{\xi}}(\tau)\,dA + B_{r6}B_{s3}\int^{A}(z-\frac{c}{2}).C_p.\rho.\overline{U}(z).R_{U'\eta}(\tau)\,dA +$$

$$B_{r6}B_{s4}[\int^{A}(z-\frac{c}{2}).C_p.\rho.\overline{U}(z).R_{U'\dot{\eta}}(\tau)\,dA + B_{r6}B_{s5}\int^{A_1}\int^{A_2} C_p^2.\rho^2.\overline{U}(z_1).\overline{U}(z_2).(z_1-\frac{c}{2}).R_{U_2'U_1'}(\tau).dA_1.dA_2 +$$

$$B_{r6}B_{s6}\int^{A_1}\int^{A_2}(z_1-\frac{c}{2})(z_2-\frac{c}{2})C_p^2.\rho^2.\overline{U}(z_1).\overline{U}(z_2).R_{U_1'U_2'}(\tau).dA_1.dA_2 \;\} \tag{48}$$

Since the wind velocity U(z,t) and the underground excitations $\xi(t)$, $\eta(t)$ are uncorrelated, the following correlation functions must have the values of zero.

$$R_{\xi U'}(\tau)=R_{\dot{\xi}U'}(\tau)=R_{\eta U'}(\tau)=R_{\dot{\eta}U'}(\tau)=0 \quad\text{and}\quad R_{U'\xi}(\tau)=R_{U'\dot{\xi}}(\tau)=R_{U'\eta}(\tau)=R_{U'\dot{\eta}}(\tau)=0 \tag{49}$$

E. The Power Spectral Density Function of the Excitations

The cross power spectral density function of the excitations with respect to general coordinates is

$$S_{Q_rQ_s}(\Omega)=F\{R_{Q_rQ_s}(\tau)\}=\int_{-\infty}^{\infty}R_{Q_rQ_s}(\tau)e^{-i\Omega\tau}d\tau \tag{50}$$

Substitute from Eqs. 48 and 49 in Eq. 50 results in

$$S_{Q_rQ_s}(\Omega)=\int_{-\infty}^{\infty}\frac{1}{\omega_r^2}\frac{1}{\omega_s^2}\;\{\;B_{r1}B_{s1}R_{\xi\xi}(\tau)+B_{r1}B_{s2}R_{\xi\dot{\xi}}(\tau)+B_{r1}B_{s3}R_{\xi\eta}(\tau)+B_{r1}B_{s4}R_{\xi\dot{\eta}}(\tau)+B_{r2}B_{s1}R_{\dot{\xi}\xi}(\tau)+B_{r2}B_{s2}R_{\dot{\xi}\dot{\xi}}(\tau)+$$

$$B_{r2}B_{s3}R_{\dot{\xi}\eta}(\tau)+B_{r2}B_{s4}R_{\dot{\xi}\dot{\eta}}(\tau)+B_{r3}B_{s1}.R_{\eta\xi}(\tau)+B_{r3}B_{s2}.R_{\eta\dot{\xi}}(\tau)+B_{r3}B_{s3}R_{\eta\eta}(\tau)+B_{r3}B_{s4}.R_{\eta\dot{\eta}}(\tau)+$$

$$B_{r4}B_{s1}R_{\dot{\eta}\xi}(\tau)+B_{r4}B_{s2}R_{\dot{\eta}\dot{\xi}}(\tau)+B_{r4}B_{s3}R_{\dot{\eta}\eta}(\tau)+B_{r4}B_{s4}R_{\dot{\eta}\dot{\eta}}(\tau)+$$

$$B_{r5}B_{s5}\int^{A}\int^{A} C_p^2.\rho^2.\overline{U}(z_1)\overline{U}(z_2).R_{U_1'U_2'}(\tau).dA_1.dA_2 +$$

$$B_{r5}B_{s6}\int^{A_1}\int^{A_2} C_p^2.\rho^2.\overline{U}(z_1)\overline{U}(z_2).(z_2-\frac{c}{2})R_{U_1'U_2'}(\tau).dA_1.dA_2 +$$

$$B_{r6}B_{s5}\int^{A_1}\int^{A_2}(z_1-\frac{c}{2})C_p^2.\rho^2.\overline{U}(z_1).\overline{U}(z_2).R_{U_1'U_2'}(\tau).dA_1.dA_2 +$$

$$B_{r6}B_{s6}\int^{A_1}\int^{A_2}(z_1-\frac{c}{2})(z_2-\frac{c}{2})C_p^2.\rho^2.\overline{U}(z_1).\overline{U}(z_2).R_{U_1'U_2'}(\tau).dA_1.dA_2 \;\}e^{-i\Omega\tau}d\tau$$

$$S_{Q_rQ_s}(\Omega)=\frac{1}{\omega_r^2}\frac{1}{\omega_s^2}\;\{\;B_{r1}B_{s1}S_{\xi\xi}(\Omega)+B_{r1}B_{s2}S_{\xi\dot{\xi}}(\Omega)+B_{r1}B_{s3}S_{\xi\eta}(\Omega)+B_{r1}B_{s4}S_{\xi\dot{\eta}}(\Omega)+B_{r2}B_{s1}S_{\dot{\xi}\xi}(\Omega)+B_{r2}B_{s2}S_{\dot{\xi}\dot{\xi}}(\Omega)+$$

$$B_{r2}B_{s3}S_{\dot{\xi}\eta}(\Omega)+B_{r2}B_{s4}S_{\dot{\xi}\dot{\eta}}(\Omega)+B_{r3}B_{s1}.S_{\eta\xi}(\Omega)+B_{r3}B_{s2}.S_{\eta\dot{\xi}}(\Omega)+B_{r3}B_{s3}S_{\eta\eta}(\Omega)+B_{r3}B_{s4}.S_{\eta\dot{\eta}}(\Omega)+$$

$$B_{r4}B_{s1}S_{\dot{\eta}\xi}(\Omega)+B_{r4}B_{s2}S_{\dot{\eta}\dot{\xi}}(\Omega)+B_{r4}B_{s3}S_{\dot{\eta}\eta}(\Omega)+B_{r4}B_{s4}S_{\dot{\eta}\dot{\eta}}(\Omega)+$$

$$B_{r5}B_{s5}\int^{A}\int^{A} C_p^2.\rho^2.\overline{U}(z_1)\overline{U}(z_2)S_{U_1U_2}(\Omega).dA_1.dA_2 +$$

$$B_{r5}B_{s6}\int^{A_1}\int^{A_2} C_p^2.\rho^2.\overline{U}(z_1)\overline{U}(z_2).(z_2-\frac{c}{2})S_{U_1U_2}(\Omega).dA_1.dA_2+$$

$$B_{r6}B_{s5}\int^{A_1}\int^{A_2}(z_1-\frac{c}{2})C_p^2.\rho^2.\overline{U}(z_1).\overline{U}(z_2).S_{U_1U_2}(\Omega).dA_1.dA_2+$$

$$B_{r6}B_{s6}\int^{A_1}\int^{A_2}(z_1-\frac{c}{2})(z_2-\frac{c}{2})C_p^2.\rho^2.\overline{U}(z_1).\overline{U}(z_2).S_{U_1U_2}(\Omega).dA_1.dA_2 \;\} \tag{51}$$

The wind velocity $\overline{U}(z)$ depends on the height of the building, according to the following equation

$$\overline{U}(z)=(\frac{z}{H})^{\alpha}\,\overline{U}(H) \tag{52}$$

Using Eq. 52 in Eq. 51

$$S_{Q_rQ_s}(\Omega)=\frac{1}{\omega_r^2}\frac{1}{\omega_s^2}\;\{\;B_{r1}B_{s1}S_{\xi\xi}(\Omega)+B_{r1}B_{s2}S_{\xi\dot{\xi}}(\Omega)+B_{r1}B_{s3}S_{\xi\eta}(\Omega)+B_{r1}B_{s4}S_{\xi\dot{\eta}}(\Omega)+B_{r2}B_{s1}S_{\dot{\xi}\xi}(\Omega)+$$
$$B_{r2}B_{s2}S_{\dot{\xi}\dot{\xi}}(\Omega)+B_{r2}B_{s3}S_{\dot{\xi}\eta}(\Omega)+B_{r2}B_{s4}S_{\dot{\xi}\dot{\eta}}(\Omega)+B_{r3}B_{s1}.S_{\eta\xi}(\Omega)+B_{r3}B_{s2}.S_{\eta\dot{\xi}}(\Omega)+B_{r3}B_{s3}S_{\eta\eta}(\Omega)+$$
$$B_{r3}B_{s4}.S_{\eta\dot{\eta}}(\Omega)+B_{r4}B_{s1}S_{\dot{\eta}\xi}(\Omega)+B_{r4}B_{s2}S_{\dot{\eta}\dot{\xi}}(\Omega)+B_{r4}B_{s3}S_{\dot{\eta}\eta}(\Omega)+B_{r4}B_{s4}S_{\dot{\eta}\dot{\eta}}(\Omega)+$$
$$B_{r5}B_{s5}.C_f^2.\rho^2.\overline{U}^2(H).S_U(\Omega)\int^{A_1}\int^{A_2}(\frac{z_1}{H})^{\alpha}(\frac{z_2}{H})^{\alpha}\gamma_{U_1U_2}(\Omega).dA_1.dA_2+$$
$$B_{r5}B_{s6}.C_f^2.\rho^2.\overline{U}^2(H).S_U(\Omega)\int^{A_1}\int^{A_2}(\frac{z_1}{H})^{\alpha}(\frac{z_2}{H})^{\alpha}(z_2-\frac{c}{2})\gamma_{U_1U_2}(\Omega).dA_1.dA_2+$$
$$B_{r6}B_{s5}.C_f^2.\rho^2.\overline{U}^2(H).S_U(\Omega)\int^{A_1}\int^{A_2}(\frac{z_1}{H})^{\alpha}(z_1-\frac{c}{2})(\frac{z_2}{H})^{\alpha}\gamma_{U_1U_2}(\Omega).dA_1.dA_2+$$
$$B_{r6}B_{s6}.C_f^2.\rho^2.\overline{U}^2(H).S_U(\Omega)\int^{A_1}\int^{A_2}(\frac{z_1}{H})^{\alpha}(z_1-\frac{c}{2})(\frac{z_2}{H})^{\alpha}(z_2-\frac{c}{2})\gamma_{U_1U_2}(\Omega).dA_1.dA_2 \tag{53}$$

These double integrals can be described as Aerodynamic Amplification Functions (Transformation Functions) are

$$\left|X_{11}(\Omega)\right|^2=\int^{A_1}\int^{A_2}(\frac{z_1}{H})^{\alpha}(\frac{z_2}{H})^{\alpha}\gamma_{U_1U_2}(\Omega).dA_1.dA_2\,,\;\left|X_{12}(\Omega)\right|^2=\int^{A_1}\int^{A_2}(\frac{z_1}{H})^{\alpha}(\frac{z_2}{H})^{\alpha}(z_2-\frac{c}{2})\gamma_{U_1U_2}(\Omega).dA_1.dA_2$$
$$\left|X_{21}(\Omega)\right|^2=\int^{A_1}\int^{A_2}(\frac{z_1}{H})^{\alpha}(z_1-\frac{c}{2})(\frac{z_2}{H})^{\alpha}\gamma_{U_1U_2}(\Omega).dA_1.dA_2\,,\qquad\} \tag{54}$$
$$\left|X_{22}(\Omega)\right|^2=\int^{A_1}\int^{A_2}(\frac{z_1}{H})^{\alpha}(z_1-\frac{c}{2})(\frac{z_2}{H})^{\alpha}(z_2-\frac{c}{2})\gamma_{U_1U_2}(\Omega).dA_1.dA_2$$

$$S_{Q_rQ_s}(\Omega)=\frac{1}{\omega_r^2}\frac{1}{\omega_s^2}\;\{\;B_{r1}B_{s1}S_{\xi\xi}(\Omega)+B_{r1}B_{s2}S_{\xi\dot{\xi}}(\Omega)+B_{r1}B_{s3}S_{\xi\eta}(\Omega)+B_{r1}B_{s4}S_{\xi\dot{\eta}}(\Omega)+$$
$$B_{r2}B_{s1}S_{\dot{\xi}\xi}(\Omega)+B_{r2}B_{s2}S_{\dot{\xi}\dot{\xi}}(\Omega)+B_{r2}B_{s3}S_{\dot{\xi}\eta}(\Omega)+B_{r2}B_{s4}S_{\dot{\xi}\dot{\eta}}(\Omega)+B_{r3}B_{s1}.S_{\eta\xi}(\Omega)+B_{r3}B_{s2}.S_{\eta\dot{\xi}}(\Omega)+$$
$$B_{r3}B_{s3}S_{\eta\eta}(\Omega)+B_{r3}B_{s4}.S_{\eta\dot{\eta}}(\Omega)+B_{r4}B_{s1}S_{\dot{\eta}\xi}(\Omega)+B_{r4}B_{s2}S_{\dot{\eta}\dot{\xi}}(\Omega)+B_{r4}B_{s3}S_{\dot{\eta}\eta}(\Omega)+B_{r4}B_{s4}S_{\dot{\eta}\dot{\eta}}(\Omega)+$$
$$C_f^2.\rho^2.\overline{U}^2(H).S_U(\Omega)\,[B_{r5}B_{s5}.\left|X_{11}(\Omega)\right|^2+B_{r5}B_{s6}.\left|X_{12}(\Omega)\right|^2+B_{r6}B_{s5}.\left|X_{21}(\Omega)\right|^2+B_{r6}B_{s6}.\left|X_{22}(\Omega)\right|^2\,]$$

Auto power spectral density function of the excitation with respect to the first general coordinates

$$S_{Q_1Q_1}(\Omega)=\frac{1}{\omega_1^2}\frac{1}{\omega_1^2}\;\{\;B_{11}B_{11}S_{\xi\xi}(\Omega)+B_{11}B_{12}S_{\xi\dot{\xi}}(\Omega)+B_{11}B_{13}S_{\xi\eta}(\Omega)+$$
$$B_{11}B_{14}S_{\xi\dot{\eta}}(\Omega)+B_{12}B_{11}S_{\dot{\xi}\xi}(\Omega)+B_{12}B_{12}S_{\dot{\xi}\dot{\xi}}(\Omega)+$$

$$B_{12}B_{13}S_{\dot{\xi}\eta}(\Omega)+B_{12}B_{14}S_{\dot{\xi}\dot{\eta}}(\Omega)+B_{13}B_{11}.S_{\eta\xi}(\Omega)+B_{13}B_{12}.S_{\eta\dot{\xi}}(\Omega)+B_{13}B_{13}S_{\eta\eta}(\Omega)+$$
$$B_{13}B_{14}.S_{\eta\dot{\eta}}(\Omega)+B_{14}B_{11}S_{\dot{\eta}\xi}(\Omega)+B_{14}B_{12}S_{\dot{\eta}\dot{\xi}}(\Omega)+B_{14}B_{13}S_{\dot{\eta}\eta}(\Omega)+B_{14}B_{14}S_{\dot{\eta}\dot{\eta}}(\Omega)+$$
$$C_f^2.\rho^2.\overline{U}^2(H).S_U(\Omega)[B_{15}B_{15}.\left|X_{11}(\Omega)\right|^2+B_{15}B_{16}.\left|X_{12}(\Omega)\right|^2+$$
$$B_{16}B_{15}.\left|X_{21}(\Omega)\right|^2+B_{16}B_{16}.\left|X_{22}(\Omega)\right|^2] \tag{55}$$

F. Complex Transformation Matrix with Respect to General Coordinates

Fourier transformation of the vibration response and excitation has the following form

$$q_n(\Omega).[-\Omega^2+i\,2D_n\omega_n\Omega+\omega_n^2]=\omega_n^2.Q_n(\Omega)$$

Where $q_n(\Omega)=H_n(\Omega).Q_n(\Omega)$ with $H_n(\Omega)=\dfrac{1}{[1-(\dfrac{\Omega}{\omega_n})^2]+i\,[2D_n(\dfrac{\Omega}{\omega_n})]}\quad,\ n=1,2,...,6$ (56)

and its absolute value is $$AHF(\Omega)=\frac{1}{[1-(\dfrac{\Omega}{\omega_n})^2]^2+[2\,D_n(\dfrac{\Omega}{\omega_n})]^2}\quad,\ n=1,2,...,6$$

G. Response Power Spectral Density Function with Respect to General Coordinates

Cross correlation functions of the response with respect to general coordinates have the form

$$R_{q_rq_s}(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}q_r(t)\,q_s(t+\tau)\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}H_r^*(\Omega)\,H_s(\Omega)\,S_{Q_rQ_s}(\Omega)\,e^{i\Omega\tau}\,d\Omega \tag{57}$$

The response power spectral density function with respect to general coordinates is

Cross: $S_{q_rq_s}(\Omega)=F\{R_{q_rq_s}(\tau)\}$ and Auto: $S_{q_n}(\Omega)=\left|H_n(\Omega)\right|^2.S_{Q_n}(\Omega)$

Auto power spectral density function for n-eigen form with respect to general coordinates is

$$S_{q_nq_n}(\Omega)=\left|H_n(\Omega)\right|^2.\frac{1}{\omega_n^2}\{\ B_{n1}B_{n1}S_{\xi\xi}(\Omega)+B_{n1}B_{n2}S_{\xi\dot{\xi}}(\Omega)+B_{n1}B_{n3}S_{\xi\eta}(\Omega)+B_{n1}B_{n4}S_{\xi\dot{\eta}}(\Omega)+B_{n2}B_{n1}S_{\dot{\xi}\xi}(\Omega)+$$
$$B_{n2}B_{n2}S_{\dot{\xi}\dot{\xi}}(\Omega)+B_{n2}B_{n3}S_{\dot{\xi}\eta}(\Omega)+B_{n2}B_{n4}S_{\dot{\xi}\dot{\eta}}(\Omega)+B_{n3}B_{n1}.S_{\eta\xi}(\Omega)+B_{n3}B_{n2}.S_{\eta\dot{\xi}}(\Omega)+$$
$$B_{n3}B_{n3}S_{\eta\eta}(\Omega)+B_{n3}B_{n4}.S_{\eta\dot{\eta}}(\Omega)+B_{n4}B_{n1}S_{\dot{\eta}\xi}(\Omega)+B_{n4}B_{n2}S_{\dot{\eta}\dot{\xi}}(\Omega)+B_{n4}B_{n3}S_{\dot{\eta}\eta}(\Omega)+B_{n4}B_{n4}S_{\dot{\eta}\dot{\eta}}(\Omega)+$$
$$C_f^2.\rho^2.\overline{U}^2(H).S_U(\Omega)[B_{n5}B_{n5}.\left|X_{11}(\Omega)\right|^2+B_{n5}B_{n6}.\left|X_{12}(\Omega)\right|^2+B_{n6}B_{n5}.\left|X_{21}(\Omega)\right|^2+B_{n6}B_{n6}.\left|X_{22}(\Omega)\right|^2]$$
$$\} \tag{58}$$

Where the mechanical amplification functions (Transformation Functions) are

$$\left|H_n(\Omega)\right|^2=\frac{1}{[1-(\dfrac{\Omega}{\omega_n})^2]^2+i\,[2D_n(\dfrac{\Omega}{\omega_n})]^2}\quad,\ n=1,2,...,6 \tag{59}$$

and the Aerodynamic Amplification Functions (Transformation Functions) are shown in Eqs. 54

H. Response Power Spectral Density Function with Respect to Original Coordinates

Cross correlation functions of the response with respect to original coordinates have the form

$$R_{X_rX_s}(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}X_r(t)\,X_s(t+\tau)\,dt\ =\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\sum_{i=1}^{6}\ \sum_{j=1}^{6}\chi_{ri}\chi_{sj}q_i(t)\,q_j(t+\tau)\,dt$$
$$=\sum_{i=1}^{6}\ \sum_{j=1}^{6}\chi_{ri}\chi_{sj}\ \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}q_i(t)\,q_j(t+\tau)\,dt=\sum_{i=1}^{6}\ \sum_{j=1}^{6}\chi_{ri}\chi_{sj}R_{q_iq_j}(\tau) \tag{60}$$

The response power spectral density function with respect to original coordinates is

Cross: $S_{X_rX_s}(\Omega)=F\{R_{X_rX_s}(\tau)\}=\sum_{i=1}^{6}\ \sum_{j=1}^{6}\chi_{ri}\chi_{sj}S_{q_iq_j}(\Omega)$ and Auto: $S_{X_n}(\Omega)=\sum_{i=1}^{6}\ \sum_{j=1}^{6}\chi_{ni}\chi_{nj}S_{q_n}(\Omega)$ (61)

$$S_{X_nX_n}(\Omega)=\sum_{i=1}^{6}\sum_{j=1}^{6}\chi_{ni}\chi_{nj}.\left|H_n(\Omega)\right|^2.\frac{1}{\omega_n^4}\{\ B_{n1}B_{n1}S_{\xi\xi}(\Omega)+B_{n1}B_{n2}S_{\xi\dot{\xi}}(\Omega)+B_{n1}B_{n3}S_{\xi\eta}(\Omega)+B_{n1}B_{n4}S_{\xi\dot{\eta}}(\Omega)+$$
$$B_{n2}B_{n1}S_{\dot{\xi}\xi}(\Omega)+B_{n2}B_{n2}S_{\dot{\xi}\dot{\xi}}(\Omega)+B_{n2}B_{n3}S_{\dot{\xi}\eta}(\Omega)+B_{n2}B_{n4}S_{\dot{\xi}\dot{\eta}}(\Omega)+B_{n3}B_{n1}.S_{\eta\xi}(\Omega)+B_{n3}B_{n2}.S_{\eta\dot{\xi}}(\Omega)+$$
$$B_{n3}B_{n3}S_{\eta\eta}(\Omega)+B_{n3}B_{n4}.S_{\eta\dot{\eta}}(\Omega)+B_{n4}B_{n1}S_{\dot{\eta}\xi}(\Omega)+B_{n4}B_{n2}S_{\dot{\eta}\dot{\xi}}(\Omega)+B_{n4}B_{n3}S_{\dot{\eta}\eta}(\Omega)+B_{n4}B_{n4}S_{\dot{\eta}\dot{\eta}}(\Omega)+$$
$$C_f^2.\rho^2.\overline{U}^2(H).S_U(\Omega)\,[B_{n5}B_{n5}.\left|X_{11}(\Omega)\right|^2+B_{n5}B_{n6}.\left|X_{12}(\Omega)\right|^2+B_{n6}B_{n5}.\left|X_{21}(\Omega)\right|^2+B_{n6}B_{n6}.\left|X_{22}(\Omega)\right|^2\,] \tag{62}$$

I. Mean Square Value Response with Respect to Original Coordinates

Mean square value of the random vibration response with respect to original coordinates can be written as

$$\psi_{X_n}^2=R_{X_nX_n}(0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{X_n}(\Omega)\,d\Omega \tag{63}$$

VI. CONCLUSIONS

This paper outlines a mathematical model describing the vibrations of high-tower buildings and its foundations with general-type equivalent passive springs and dampers, rigid bodies, and some ideal constraints under the effect of randomly fluctuating wind loads and the excitation of earthquake ground motions. Two derivation methods of the equivalent system's differential equations have been considered, namely D'alembert's principle and Lagrange's method, which verified the acceptability of the developed equations of motion. Following conclusions can be withdrawn:

- The mathematical model with 6 degrees of freedom presented in the present paper can be used to investigate the effect of both wind and earthquakes loading.
- Analytical solution of the free vibrations of tall building and its foundation using the general modal analysis method has been performed.
- Analytical solution of forced vibrations of tall building and its foundation has been developed, through the correlation function (time domain) and the power spectral density function (frequency domain) of system response with respect to general and also original coordinates.
- Without wind and earthquakes, structures – particularly large ones – it would probably be a lot easier to design and cheaper.
- Random vibrations of building's foundation subjected to seismic excitations of earthquake ground motions and also randomly fluctuating wind pressure fields acting on a building surface are analyzed.

NOMENCLATURE

C_p - Aerodynamic pressure factor (-)

E - Kinetic energy of the system (J)

E_d-Soil dynamic modulus of elasticity (kp/m^3)

F_{1H}, F_{1V} - Spring and damping forces at C or E in horizontal and vertical direction (kp)

F_{2H}, F_{2V} - Spring and damping forces at D or F in horizontal and vertical direction (kp)

F_{EH}, F_{EV} - Spring and damping forces at s_1 in horizontal and vertical direction due to earthquake effect (kp)

$H(i\Omega)$ - imaginary transformation function (-)

J_1, J_2 - Mass moment of Inertia of foundation with its accompanied vibrated soil and tall building (kg.s^2.m)

J_F, J_S - Mass moment of Inertia of foundation and accompanied vibrated soil with it (kg.s^2.m)

k_{EH}, k_{EV} - Linear horizontal and vertical equivalent spring stiffness of earth (kp/m)

k_{EK} - Rotational equivalent spring stiffness of earth (kp.m/rad)

k_H, k_V -Linear horizontal and vertical equivalent spring stiffness of building-foundation connection (kp/m)

L - Lagrangian function (-)

m_1, m_2 -Total mass of foundation with its accompanied vibrated soil (m_F+ m_S) and tall building (kg)

m_F, m_S - Foundation and Vibrating soil mass (kg)

$M_W(t)$ - Total turbulent wind moment as a function of time (kp.m)

q_K - General coordinates $z_1^*, y_1^*, \varphi_1^*, z_2^*, y_2^*, and$ φ_2^* (m, m, rad, m, m, rad)

$R_{Q_iQ_j}(\tau)$ - Cross correlation function of the excitations (m^2)

$R_{q_rq_s}(\tau), R_{X_rX_s}(\tau)$ - Cross correlation function of response with respect to general and original coordinates (m^2)

$\Re$ - Rayleigh's dissipation function (kp.m/s)

r_b -Vertical embedding damping constant : the damping constant of radiation (kp.s/m^3)

r_{EK}- Rotational equivalent damping coefficient of earth (kp.m.s/rad)

r_{EH}, r_{EV} - Linear horizontal and vertical equivalent damping coefficient of earth (kp.s/m)

r_H, r_V - Linear horizontal, vertical equivalent damping coefficient of building-foundation connection (kp.s/m)

r_S - Damping coefficient of the elastic soil bed (kp.s/m^3)

s_1, s_2 - Centre of gravity of the foundation and tall building (-)

$S_{q_iq_i}(\Omega), S_{q_rq_s}(\Omega)$ - Auto and cross power spectral density function of response w.r.t. general coordinates (m^2.s/rad)

$S_{Q_iQ_i}(\Omega), S_{Q_rQ_s}(\Omega)$ - Auto and cross power spectral density function of excitations (m^2.s/rad)

$S_{X_nX_n}(\Omega), S_{X_rX_s}(\Omega)$ - Auto and cross power spectral density function of response w.r.t. original coordinates (m^2.s/rad)

t - Time (s)

T_{EK} - Spring and damping torques about s_1 in rotational direction (kp.m)

U - Potential energy of the system (J)

$\overline{U}(H)$ - Average wind velocity along the building height H (m/s)

$U_y(t), U_z(t)$ -Random displacement excitation of earthquake in horizontal and vertical direction (m)

$U(z,t)$ - Wind speed as a function of space and time (m/s)

$\overline{U}(z)$ - Constant part of wind speed as a function of space (m/s)

$U'(z,t)$ - Turbulent part of wind speed as a function of space and time (m/s)

V_S - Vertical wave velocity (m/s)

$W(t)$ - Total turbulent wind force in y^*-direction as a function of time (kp)

$W(z,t)$ - Wind load as a function of space and time (kp)

$\overline{W}(z)$ - Constant part of wind load as a function of space (kp)

$W'(z,t)$ - Turbulent part of wind load as a function of space and time (kp)

$\underline{\hat{x}}$ - Amplitude of exponential solution of motion differential equations (m)

$y_o^*(t), z_o^*(t)$ - Displacement of point O in the direction of y_o^* and $z_o^* - axis$ (m)

$y_1(\tau), z_1(\tau), \varphi_1(\tau), y_2(\tau), z_2(\tau), \varphi_2(\tau)$ - non-dimensional Displacements (-)

$y_1'(\tau), z_1'(\tau), \varphi_1'(\tau), y_2'(\tau), z_2'(\tau), \varphi_2'(\tau)$ - non-dimensional velocities (-)

$y_1''(\tau), z_1''(\tau), \varphi_1''(\tau), y_2''(\tau), z_2''(\tau), \varphi_2''(\tau)$ - non-dimensional accelerations (-)

$y_1^*(t), z_1^*(t)$ - Displacement of gravity centre s_1 of foundation in y_1^* and z_1^* - axis (m)

$y_2^*(t), z_2^*(t)$ - Displacement of gravity centre s_2 of high tower building in y_2^* and z_2^* - axis (m)

$y_C^*(t), z_C^*(t)$ - Displacement of point C in the direction of y_C^* and $z_C^* - \text{axis}$ (m)

$\dot{y}_C^*(t), \dot{z}_C^*(t)$ - Velocity of point C in the direction of y_C^* and $z_C^* - axis$ (m/s)

$\ddot{y}_C^*(t), \ddot{z}_C^*(t)$ - Acceleration of point C in the direction of y_C^* and $z_C^* - axis$ (m/s^2)

$y_D^*(t), z_D^*(t)$ - Displacement of point D in the direction of y_D^* and $z_D^* - axis$ (m)

$\dot{y}_D^*(t), \dot{z}_D^*(t)$ - Velocity of point D in the direction of y_D^* and $z_D^* - axis$ (m/s)

$\ddot{y}_D^*(t), \ddot{z}_D^*(t)$ - Acceleration of point D in the direction of y_D^* and $z_D^* - axis$ (m/s^2)

$y_E^*(t), z_E^*(t)$ - Displacement of point E in the direction of y_E^* and $z_E^* - axis$ (m)

$\dot{y}_E^*(t), \dot{z}_E^*(t)$ - Velocity of point E in the direction of y_E^* and $z_E^* - \text{axis}$ (m/s)

$\ddot{y}_E^*(t), \ddot{z}_E^*(t)$ - Acceleration of point E in the direction of y_E^* and $z_E^* - axis$ (m/s^2)

$y_F^*(t), z_F^*(t)$ - Displacement of point F in the direction of y_F^* and $z_F^* - axis$ (m)

$\dot{y}_F^*(t), \dot{z}_F^*(t)$ - Velocity of point F in the direction of y_F^* and $z_F^* - axis$ (m/s)

$\ddot{y}_F^*(t), \ddot{z}_F^*(t)$ - Acceleration of point F in the direction of y_F^* and $z_F^* - axis$ (m/s^2)

α - Profile constant (-)

γ_B - Specific weight of the high tower building (kp/m^3)

ρ, ρ_1, and ρ_2 - Density of air, foundation, and high tower building respectively (kg/m^3)

τ - non-dimensional time [-]

$\varphi_o^*(t)$, $\varphi_1^*(t)$, $\varphi_2^*(t)$ - Angular displacements about x_o^*, x_1^*, and $x_2^* - axis$ [rad]

$\varphi_o(t)$, $\varphi_1(t)$, $\varphi_2(t)$ - non-dimensional angular displacement about x_o^*, x_1^*, and $x_2^* - axis$ [-]

APPENDIX

$m_F = \rho_F . V_F$, Foundation weight $= W_F = m_F . g$, Lorenz, H. (1955) calculated the weight of the accompanied vibrating soil with the foundation using the equation $W_S = f.A_F^{(4/3)} = [0.835].[a.b]^{(4/3)}$ ton, $m_S = W_S / g$ kg.

$$m_1 = m_F + m_S, \; J_1 = J_F + m_F . l_f^2 + J_S + m_S . l_S^2,$$

$$l_F = \frac{m_S}{m_F} . l_S = \frac{m_S}{m_F}\left(\frac{d+h}{2} - l_F\right) = \frac{[m_S / m_F][(d+h)/2]}{[1 + (m_S / m_F)]}$$

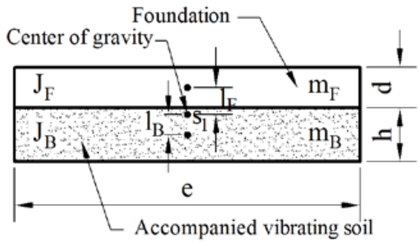

Fig. 4 Foundation with its accompanied vibrating toned sand

$$h = \frac{W_S}{A_F.r_S}, \; l_S = [(d+h)/2 - l_F], \; J_F = m_F.[(d^2+e^2)/12],$$

$$J_S = m_S.[(h^2+e^2)/12]$$

Vertical embedding damping constant: the damping constant of radiation is $r_b = E_d / V_S$

Mass of the high tower: the density of high tower can be assumed as 1/10 of that of the foundation, i.e. $\rho_2 = \rho_1 / 10$

$$m_2 = \rho_2.V_2 \;,\; W_2 = m_2.g \;,\; J_2 = m_2.[(b^2+c^2)/12]$$

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